Lecture Notes for 10/10/2023

4.4 Basis and dimension

First, let us do a brief review.

Definition of a span and a spanning set: Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be k given vectors in $\mathbb{R}^n$. The span of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$, $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_\text{k})$, is the set that contains all possible linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$. $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_\text{k})$ is a subspace of $\mathbb{R}^n$ and $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ is called a spanning set of this subspace.

Example 1. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is what? What about the span of $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$?

$$a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \dim(\text{Span}(v_1, v_2)) = 2$$

$$a_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 3a_1 \\ -4a_2 \end{bmatrix}$$

Example 2. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is the same as the span of $\mathbf{v}_1$ since $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$. So $\text{Span}(\mathbf{v}_1, \mathbf{v}_2) \neq \mathbb{R}^2$. Why?

$$a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_2 \\ -(a_1 + 2a_2) \end{bmatrix}$$

$$\times \begin{bmatrix} 2 \\ 3 \end{bmatrix} ? \qquad \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Quiz Question 1. Which of the following statement is correct?

A. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is $\mathbb{R}^2$;

B. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is $\mathbb{R}^3$;

C. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is $\mathbb{R}^3$;

D. The span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not $\mathbb{R}^3$.

Definition of linear independence and dependence: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are (called) linearly independent if the only solution to the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

is $a_1 = a_2 = \cdots = a_k = 0$. If on the other hand, this equation has (non-trivial) solutions, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are (said to be) linearly dependent.

Example 1. Determine whether

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

?

$$\begin{bmatrix} 2 & 1 & 5 \\ -1 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -1 & -2 \\ 2 & 1 & 5 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{2R_1+R_2 \to R_2} \begin{bmatrix} -1 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} -1 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so}$$

Example 2. Determine the values of c so that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \qquad c \neq -\frac{2}{5}$$

are linearly independent.

$$\left| \begin{matrix} c & 1 & 0 \\ 1 & -1 & 3 \\ 0 & 1 & 2 \end{matrix} \right. \begin{matrix} c & 1 \\ 1 & -1 \\ 0 & 1 \end{matrix}$$

$$-2c - 3c - 2 = 0$$

$$-5c = 2 \qquad c = -\frac{2}{5}$$

Quiz Question 2. Determine the value of c so that the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} c \\ 5 \\ -2 \end{bmatrix}$$

are linearly DEPENDENT.

A. 1 B. 12 C. -12 D. 22

$$\begin{array}{ccccc} 1 & 3 & c & 1 & 3 \\ 0 & -1 & 5 & 0 & -1 \\ 1 & 1 & -2 & 1 & 1 \end{array}$$

$2 + 15 - (-c) - 5 = 0$

Definition of a basis. A spanning set of a vector space is called a basis of the vector space if it is linearly independent. Note a vector space can be a subspace in $\mathbb{R}^n$, not necessary the entire $\mathbb{R}^n$.

Example: $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$ is a basis of $\mathbb{R}^n$ (called the standard basis).

$$W = \text{span}(\underbrace{v_1, v_2, \cdots, v_K}_{\text{basis}})$$

The following theorem is quite important in the context of this section.

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ and $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m$ are both bases of the same vector space V, then $m = n$.

Because of this theorem, we can define the dimension of a vector space V as the number of vectors in any of its bases.

It is also important to note the following theorem.

Theorem. If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is a basis of the vector space V, then any vector $v \in V$ can be written a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ and this linear combination is unique. Say

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

We say that $c_1, c_2, ..., c_n$ are the *coordinates* of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$.

$$\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c_1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1,\ \cancel{v_2},\ \cancel{v_3},\ v_4,\ v_5$$

If our vector space is $\mathbb{R}^n$ then obviously its dimension is n since we know its standard basis contains n vectors. But what if our vector space is a subspace of $\mathbb{R}^n$ which is spanned by some vectors?

Answer: using Gaussian elimination method to find a basis within a known spanning set of the subspace.

$$\dim(W) = 2$$

Example. Find the dimension of the subspace W of $\mathbb{R}^3$ that is the span of

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \cancel{\begin{bmatrix} 3 \\ -1 \\ -11 \end{bmatrix}}. \right\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ -3 & 2 & 1 & -11 \end{bmatrix} \xrightarrow{3R_1 + R_3 \Rightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$$\xrightarrow{-2R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quiz Question 3. Find the dimension of the subspace in $\mathbb{R}^4$ whose spanning set contains the following vectors:

$$\begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 4 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}.$$

A. 1   B. 2   C. 3   D. 4

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 4 & -2 & 4 & 2 \\ 0 & 0 & -3 & -1 \end{bmatrix} \xrightarrow{-2R_1 + R_3 \to R_3} \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & \cancel{6} & \cancel{2} \\ 0 & 0 & \cancel{-3} & \cancel{-1} \end{bmatrix}$$

Theorem. Let A be a matrix of size $m \times n$. The set of all solutions to a homogeneous linear equation system (written in matrix form) $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Why?

Let W be the subspace given by the above theorem. How do we determine its dimension (or find a basis of it)?

Example.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\ \ x_2 \qquad\ x_4 \end{array}$$

$x_1 = -2x_2 + 3x_4$

$x_1 + 2x_2 - 3x_4 = 0$

$x_3 - x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3x_4 \\ 1 \cdot x_2 + 0 \cdot x_4 \\ 0 \cdot x_2 + 1 \cdot x_4 \\ 0 \cdot x_2 + 1 \cdot x_4 \end{bmatrix} = \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + \underbrace{x_4 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{v_2}$$

Quiz Question 4. Let W be the subspace of $\mathbb{R}^7$ that consists of the solutions to the equation system $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Find the dimension of W.

A. 1 B. 2 C. 3 D. 4