$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \vec{0}$

If $a_1 = \cdots = a_k = 0$ is the only solution, then $v_1, \ldots, v_k$ independent.

Lecture Notes for 10/12/2023

5.1 General vector spaces

5.2 Subspaces

Field: a system similar to the set of real numbers with two operations (addition and multiplications) defined, and behaves like the real numbers. (Do not overburden yourself with this: we will only be using real numbers in this course.)

If we take out the "geometric meaning" of a vector in $\mathbb{R}^n$, then what we are looking at is just a matrix of size $n \times 1$. The vector addition and scalar multiplication in $\mathbb{R}^n$ are just the matrix addition and matrix scalar multiplication. So if we replace the vectors in $\mathbb{R}^n$ by matrices of the same size, say by the set of all 2×2 matrices, then Conditions (1) to (10) that we used to define the vector space $\mathbb{R}^n$ would still hold. That is, the set of all 2×2 matrices with real number entries behaves just like the vector space $\mathbb{R}^4$ under the matrix addition and scalar multiplication, so there is no reason why we cannot think it as a vector space. We just have to ignore its geometric meaning. We would call such a vector space a *general vector space* to distinguish it from a Euclidean vector space $\mathbb{R}^n$. (Or we can call it an "abstract vector space" like in some literature.)

Definition 5.1.2: General vector spaces.

A ***vector space*** V over a field F is a set that satisfies a list of properties under two binary operations, vector addition and scalar multiplication. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars a and b in F. Then,

- Closure under vector addition: $\mathbf{u} + \mathbf{v} \in V$
- Closure under scalar multiplication: $a\mathbf{u} \in V$
- Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- Additive identity: $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- Additive inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- Associativity of scalar multiplication: $a(b\mathbf{u}) = (ab)\mathbf{u}$
- Scalar identity: $1 \cdot \mathbf{u} = \mathbf{u}$
- Distributivity of scalars over vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- Distributivity of vectors over scalar addition: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

$\underline{\underline{\mathbb{R}}}_{n \times m}$

$\mathbb{R}_{3 \times 4}$

$\mathbb{R}^n$

More examples of "general vector spaces".

$\mathbb{R}_{3\times 2}$: the set of all 3×2 matrices with real number entries. What about the set that contains all 3×2 AND all 2×2 matrices with real entries? Or the set of all 2×2 matrices with integer entries?

$\mathbb{R}_{2\times 2} \cup \mathbb{R}_{3\times 2}$ is not

$\mathcal{P}_n$: the set of all polynomials up to degree n with real number coefficients.

$\mathcal{P}_2$: $2+x$, $1-x+5x^2$, 7, 0

$3x-4x^2$ ...

$(3x-4x^2)+(1-x+5x^2)$

$= 1+2x+x^2$

$5(3x-4x^2)$

Quiz Question 1. Let U be the set that contains all 1×5 matrices with real entries, V be the set that contains all 3×3 matrices with real entries, and W be $\mathbb{R}^4$, which of the following statement is NOT true?

$U = \mathbb{R}_{1\times5}$ $\quad V = \mathbb{R}_{3\times3}$

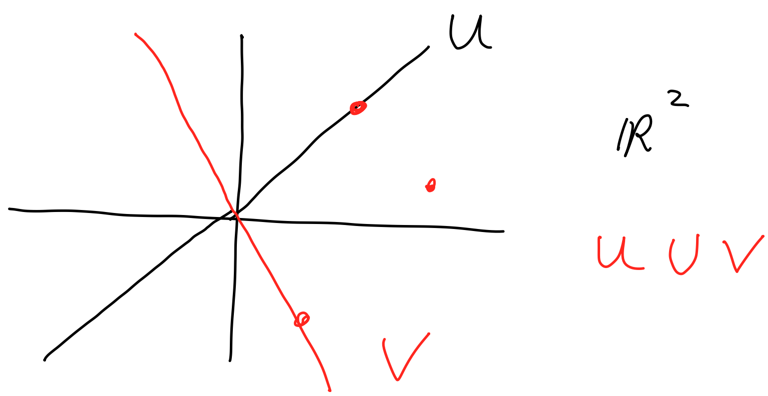
$W = \mathbb{R}_{4\times1} = \mathbb{R}^4$

A. U is a vector space;

B. V is a vector space;

C. The union of U, V and W is a vector space.

D. Each of U, V and W is a vector space.



And examples of subspaces of these general vector spaces.

W is the set of all 3×3 diagonal matrices. with real entries.

$W \subseteq \mathbb{R}_{3\times 3}$ W is closed under addition and scalar multiplication.

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} + \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & & \\ & a_2+b_2 & \\ & & a_3+b_3 \end{bmatrix} \in W$$

W is the set of all 2×2 matrices such that the sum of its entries is zero.

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \notin W \qquad \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \in W$$

if $c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in W$, $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in W$

$c(a_{11}+a_{12}+a_{21}+a_{22}) = 0$

$b_{11}+b_{12}+b_{21}+b_{22} = 0$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \in W$$

$W = \{a_1x + a_2x^2 : \ a_1, \ a_2 \in \mathbb{R}\}$ (which is a subset of $\mathcal{P}_2$).

$a_0 = 0$

$a_0 + a_1 x + a_2 x^2$

$$\begin{array}{r} a_1x + a_2x^2 \\ + \; b_1x + b_2x^2 \end{array} = (a_1+b_1)x + (a_2+b_2)x^2$$

Examples of subsets that are not subspaces.

The set of all 2×2 matrices whose $\boxed{\textit{traces}}$ are integers.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{trace}(A) = a_{11} + a_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The set of all 2×2 matrices whose determinants are zero.

$$|A| = 0 \qquad \det(cA) = c^2 \det(A) = 0$$

$$\underbrace{\overset{\in W}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} + \overset{\in W}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}} \notin W$$

The set of all degree 2 (not less or equal to two!) polynomials with real number coefficients.

$$\mathcal{P}_2$$

$$a_0 + a_1 x + a_2 x^2$$

$$a_2 \neq 0$$

Quiz Question 2. Identify which of the following is a vector space.

A. The set of all 3×3 matrices with integer traces;

B. The set of all your textbooks;

C. The set of all 3×3 matrices with real number entries whose determinants are zero;

D. The set of all polynomials of the form $ax^3 + b$ with a and b being any real numbers.

Linear dependence/independence, basis. Examples.

How do we determine whether a set of vectors is linearly independent in a general vector space? We will do this by using a STANDARD basis of the general vector space.

$0 + 0 \cdot x + 0 \cdot x^2$

Standard basis for $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, etc.

$1+2x,\ 3x-2x^2,\ 4x+7x^2 \in \mathcal{P}_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 0 & -2 & 7 \end{bmatrix}$$

are these linearly independent?

$a_1(1+2x) + a_2(3x-2x^2) + a_3(4x+7x^2) = 0$

$$\begin{cases} a_1 = 0 \\ 2a_1 + 3a_2 + 4a_3 = 0 \\ -2a_2 + 7a_3 = 0 \end{cases}$$

$\mathcal{P}_2$: $1,\ x,\ x^2$

$5+3x-7x^2$

$3-5x+4x^2 \longrightarrow \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$ $\qquad \begin{bmatrix} 5 \\ 3 \\ -7 \end{bmatrix}$

$\mathcal{P}_3$: $1,\ x,\ x^2,\ x^3$

Example. $\text{Span}\left(1+x+x^2,\ 1-2x+3x^3,\ x+5x^2,\ 5-7x^3\right)$

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 3 & 0 & -7 \end{bmatrix}$$

$\mathbb{R}^3$, $\mathbb{R}^4$

Standard basis for $\mathbb{R}_{n\times m}$.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\mathbb{R}_{2\times 2}$

$$\left\{ \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right.$$

$$\begin{bmatrix}1 & -3\\4 & 7\end{bmatrix} \longrightarrow \begin{bmatrix}1\\-3\\4\\7\end{bmatrix}$$

Example. Determine whether $1-2x+3x^2$, $x-x^2$ and $3-8x+11x^2$ are linearly independent.

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -8 \\ 3 & -1 & 11 \end{bmatrix}$$

Repeat the above for $a_1\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, $+a_2\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, $+a_3\begin{bmatrix} 5 & -1 \\ 2 & 10 \end{bmatrix}$, $+a_4\begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$. $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ -3 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 4 & 10 & -1 \end{bmatrix}$$

Quiz Question 3. Determine whether the vectors $1-2x$, x^2 and $x+x^2$ of $\mathcal{P}_2$ are linearly independent.

A. They are dependent today, but will become independent tomorrow;

B. They are linearly independent; $\det\begin{vmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \neq 0$

C. They are linearly dependent;

D. I pray that this kind of problems will not be on the test.

Example. Let W be the subspace of $\mathcal{P}_3$ with a spanning set consisting of $1-3x+x^2$, $2-6x+2x^2$, $x+2x^2+x^3$ and $1-x+4x^2+2x^3$. Find a basis for W.

Quiz Question 4. If the vectors $2-3x$, x^2 and $x+x^2$ are linearly independent, then (choose the correct statement):

A. The matrix $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ must have none-zero determinant;

B. The matrix $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ must have zero determinant;

C. The echelon form of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ must have a non-pivot column;

D. The equation $c_1(2-3x)+c_2x^2+c_3(x+x^2)=0$ must have non-zero solutions.