Lecture Notes for 10/19/2023

5.3 Coordinatization (Continued)

5.4 Four fundamental subspaces

Continue from 5.3: Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$. Since this is also a basis, we can also write $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$ as a (unique) linear combination of the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$:

$$\overset{2}{x_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \overset{-3}{x_2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \overset{1}{x_3} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$$

$M_B X = V$

The matrix form of this equation system is $M_{\mathcal{B}}\mathbf{x} = \mathbf{v}$, where

$$M_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}.$$

The matrix $M_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible: $M_{\mathcal{B}}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ so

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{[v]_B} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_B$$

$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_B$

We call this vector the *coordinate vector* of $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$ under the basis $\mathcal{B}$ and denote it by $[\mathbf{v}]_{\mathcal{B}}$:

$$[\mathbf{v}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_B$$

M_B^{-1} transition

Now let us find $[\mathbf{u}]_{\mathcal{B}}$ for $\mathbf{u} = \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix}$ and $[\mathbf{w}]_{\mathcal{B}}$ for $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$:

$$[\mathbf{u}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{u} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -8 \\ 6 \\ 6 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \\ 3 \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{w}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{w} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{B}}$$

We can generalize this concept to an arbitrary basis of $\mathbb{R}^n$:

Let $\mathcal{B} = \{\underbrace{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n}\}$ be a basis of $\mathbb{R}^n$ with the vectors in it given a fixed order as shown. Keep in mind each $\mathbf{u}_j$ in the basis is an $n \times 1$ column matrix. Since $\mathcal{B}$ is a basis, it is a spanning set of $\mathbb{R}^n$ hence every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written as a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots c_n\mathbf{u}_n = \mathbf{v}.$$

$M_B = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$

We shall call the vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ the *coordinate* vector of $\mathbf{v}$ with respect to the basis $\mathcal{B}$ and will use the notation $[\mathbf{v}]_{\mathcal{B}}$ for it.

M_B^{-1}

Note that to find the coordinate vector $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ of $\mathbf{v}$ means to solve the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots c_n\mathbf{u}_n = \mathbf{v},$$

which has the matrix form $M_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \mathbf{v}$, where $M_{\mathcal{B}}$ is the matrix with $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ as its columns (in that order) and $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. $M_{\mathcal{B}}$ is invertible (why?), so

$$[\mathbf{v}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{v}.$$

Keep in mind that $M_{\mathcal{B}}$ is the matrix with the vectors in $\mathcal{B}$ as its columns, which is why we denote it by $M_{\mathcal{B}}$.

Example 1. Let $\mathcal{B} = \{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \}$ be a basis of $\mathbb{R}^2$. Find $M_{\mathcal{B}}$ and $M_{\mathcal{B}}^{-1}$, then use that to find $[\mathbf{v}]_{\mathcal{B}}$ where $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

$$M_B = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, \quad M_B^{-1} = \frac{1}{1}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$[v]_B = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 11 \end{bmatrix}_B$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 17\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 11\begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

Example 2. Repeat Example 1 for $\mathcal{B} = \{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

$M_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$, and $M_{\mathcal{B}}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 2 \\ 4 & 2 & -1 \end{bmatrix}$ $\quad 1 - 2 + 2 = 1$

$$[v]_B = \frac{1}{5}\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 2 \\ 4 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 7/5 \\ 4/5 \end{bmatrix}_B$$

$2 \cdot 1 + 1 \cdot 1 + 2 \cdot 2$

$2 + 1 + 4$

$4 \cdot 1 + 2 \cdot 1 + (-1) \cdot 2 = 4 + 2 - 2 = 4$

Quiz Question 1. Repeat the last example with the same basis $\mathcal{B}$, but we are looking for the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ for a different vector $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

A. $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$; B. $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$; C. $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix}$; D. $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$M_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}, \ M_{\mathcal{B}}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 2 \\ 4 & 2 & -1 \end{bmatrix}$$

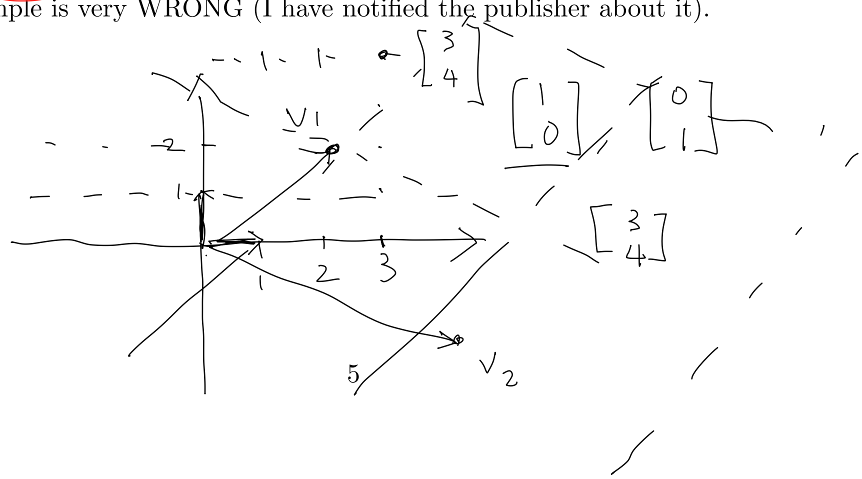
We see the advantage of this approach from this quiz question: once we find the inverse matrix $M_{\mathcal{B}}^{-1}$ (which we only need to do it one time), finding the coordinate vector of any vector $\mathbf{u}$ in the future is just a simple matrix multiplication:

$$[\mathbf{u}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{u}$$

$M_{\mathcal{B}}^{-1}$ $P_{\mathcal{B}}$

In our textbook, the inverse matrix $M_{\mathcal{B}}^{-1}$ is denoted by another notation $P_{\mathcal{B}}$ and is referred to as the *transition matrix* from the standard basis $\mathcal{E}$ to the basis $\mathcal{B}$.

The geometric meaning of basis change is to change from the rectangular coordinate system (grid) to one that may no longer be rectangular. See the example 5.3.1: Understanding coordinate systems in the book. HOWEVER, this example is very WRONG (I have notified the publisher about it).



In the above, we know the coordinate vector of a vector $\mathbf{u}$ relative to the standard basis $\mathcal{E}$ and the formula $[\mathbf{u}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{u}$ allows us to obtain the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ of $\mathbf{u}$ relative to $\mathcal{B}$. Sometimes, however, we already know the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ of $\mathbf{u}$ relative to $\mathcal{B}$ to begin with, and we need to switch to yet another basis $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n\}$ so we need to find $[\mathbf{u}]_{\mathcal{C}}$. How do we do that?

$u \to M_B^{-1} \cdot u = [u]_B \qquad B \longrightarrow C \ ?$

$P_{B\to C}$

$M_B^{-1} u = [u]_B$

$u = M_B [u]_B$

$[\mathbf{u}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{u}$, $\quad [\mathbf{u}]_{\mathcal{C}} = M_{\mathcal{C}}^{-1}\mathbf{u}$, so $\mathbf{u} = M_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}}$ and $[\mathbf{u}]_{\mathcal{C}} = M_{\mathcal{C}}^{-1}M_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}}$: we know $[\mathbf{u}]_{\mathcal{B}}$ to begin with, and we know $M_{\mathcal{B}}$ and $M_{\mathcal{C}}$ (matrices obtained by using vectors in $\mathcal{B}$ and $\mathcal{C}$ as columns), so this only involves finding the inverse of $M_{\mathcal{C}}$ followed by a matrix multiplication. $M_{\mathcal{C}}^{-1}M_{\mathcal{B}} = P_{\mathcal{B}\to\mathcal{C}}$ is called the change of coordinates matrix. It may help to remember that if we are changing from basis $\mathcal{B}$ to basis $\mathcal{C}$, then in the basis change formula, $M_{\mathcal{C}}^{-1}$ is the inverse of the matrix formed by the vectors in the basis we are changing to.

$[u]_B = M_B^{-1} \cdot u$

$u \longrightarrow [u]_C \qquad M_C^{-1} \cdot u$

Example. Given that $\mathcal{B} = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\}$ and $\mathcal{C} = \{\begin{bmatrix}2\\-3\end{bmatrix}, \begin{bmatrix}-3\\5\end{bmatrix}\}$ are both bases of $\mathbb{R}^2$. Find the change of coordinates matrix from basis $\mathcal{B}$ to basis $\mathcal{C}$.

The matrix is

$$M_{\mathcal{C}}^{-1}M_{\mathcal{B}} = \begin{bmatrix}2 & -3\\-3 & 5\end{bmatrix}^{-1}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}5 & 3\\3 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}5 & 13\\3 & 8\end{bmatrix}$$

So if we have $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix}1\\-3\end{bmatrix}_B$, then $[\mathbf{u}]_{\mathcal{C}} = \begin{bmatrix}5 & 13\\3 & 8\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}_B = \begin{bmatrix}-34\\-21\end{bmatrix}_C$

$$1 \cdot \begin{bmatrix}1\\0\end{bmatrix} - 3\begin{bmatrix}2\\1\end{bmatrix} = -34\begin{bmatrix}2\\-3\end{bmatrix} + (-21)\begin{bmatrix}-3\\5\end{bmatrix}$$

Quiz Question 2. Given that $\mathcal{B} = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\}$ and $\mathcal{C} = \{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ are both bases of $\mathbb{R}^2$. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

A. $\begin{bmatrix}1 & 2\\2 & 5\end{bmatrix}$ B. $\begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}$ C. $\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ D. $\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$ $M_C^{-1} M_B$

$\begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}^{-1} \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$

Let us now consider the same question, but for general vector spaces.

Here, finding $[\mathbf{v}]_{\mathcal{B}}$ is just solving the equation $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ where $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$. We can write down a basis change formula as before: $[\mathbf{v}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}\mathbf{v}$ with the understanding that all the vectors here are the coordinate vectors under a standard basis.

Example. Find the transition matrix from the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_2$ to the basis $\mathcal{B} = \{1 - x, x + x^2, 2x - x^2\}$. Using it to find the coordinate vector for the vector $\mathbf{v} = 1 + 9x + x^2$ under $\mathcal{B}$.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

$$M_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \qquad M_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

So

$$[\mathbf{v}]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}_{\mathcal{B}}.$$

$$1 + 9x + x^2 = 1 \cdot (1 - x) + 4(x + x^2) + 3(2x - x^2)$$

Quiz Question 3. Find the transition matrix from the standard basis $\{1, x\}$ of $\mathcal{P}_1$ to the basis $\mathcal{B} = \{2 + 3x, 1 + 2x\}$.

M_B $\qquad$ M_B^{-1}

A. $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ C. $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ D. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Section 5.4.

$A_{m\times n}$ $\quad v_1 \in \mathbb{R}^m$ $\quad$ left null

Let $A_{m\times n}$ be the matrix whose columns are $\mathbf{v}_1$, ..., $\mathbf{v}_n$. Then we call the span of $\mathbf{v}_1$, ..., $\mathbf{v}_n$ the *column space* of A and write it as col(A). This is just a different name for Span($\mathbf{v}_1, ..., \mathbf{v}_n$). Furthermore, a linear combination

$A^T x = 0$

$\boxed{Ax = 0}$ $\quad m$

rank (A)

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$$

dim(col(A)) + dim(null(A)) = n

can be written as $A\mathbf{x}$ with $A = (\mathbf{v}_1\ \mathbf{v}_2\ ...\mathbf{v}_n)$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ null null (A)

rank(A^T) = rank (A)

rank(A^T) + ~~dim~~ dim(left null) $= m$

col$\boxed{(A^T)}$

dim(col(A)) is also defined as the *rank* of A and written as rank(A).

We can find rank(A) by using row operations to find an echelon form of A and then count the number of pivots. Another vector space is the *null* space of A, denoted by null(A), which is the set containing all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. The dimension of $\boxed{\text{null}(A)}$ equals the number of free variables. Thus dim(col(A)) + dim(null(A)) = n (the number of columns in A).

Similarly we can define the *row* space of A, which is the span of the columns of A^T, and the *left null* space of A, which is the solution set of the equation $A^T\mathbf{x} = \mathbf{0}$. dim(col($A^T$)) + dim(null($A^T$)) = m. Additionally, we have

$$\dim(\text{col}(A)) = \dim(\text{col}(A^T)) \text{ (or rank}(A) = \text{rank}(A^T)).$$

$n = 5$

Example. If the matrix A reduced to the echelon form $\begin{bmatrix} 1 & 1 & 3 & -1 & 4 \\ 0 & 0 & 2 & 7 & -3 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

find dim(col(A)), $\underbrace{\text{dim(col}(A^T))}_{}$ 3, dim(null(A)) 2, dim(null(A^T)) 1

dim(col(A)) = 3

$A_{9\times 7}$

rank(A) = 3

~~row~~ dim(col(A^T)) = 3

dim(null (A)) = 4

dim(null (A^T)) = 6

Quiz Question 4. Given that A has size 7×5 and that the rank of A is 3, then what is the dimension of null(A) and what is the dimension of null(A^T)?

A. $\dim(\text{null}(A)) = 2$ and $\dim(\text{null}(A^T)) = 4$;

B. $\dim(\text{null}(A)) = 3$ and $\dim(\text{null}(A^T)) = 5$;

C. $\dim(\text{null}(A)) = 5$ and $\dim(\text{null}(A^T)) = 7$;;

D. $\dim(\text{null}(A)) = 2$ and $\dim(\text{null}(A^T)) = 2$.