Lecture Notes for 10/3/2023

4.1 Vector spaces and subspaces

The Euclidean vector spaces are the generalizations of the 1, 2 and 3 dimensional spaces that we are familiar with:

$\mathbf{R}$: the set of all real numbers;

$\mathbf{R}^2$: the set of all points in a plane with a rectangular coordinate system, but with its coordinates written in the form of a column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$;

$\mathbf{R}^3$: the set of all points in a 3-dimensional space with a rectangular coordinate system, also with its coordinates written in the form of a column vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$;

$\mathbf{R}^4$: the set of all points in a 4-dimensional space with a "rectangular coordinate system", also with its coordinates written in the form of a column vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$;

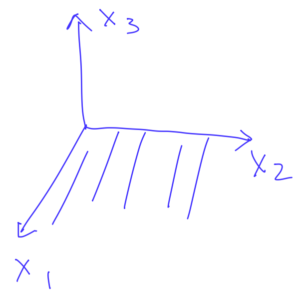
...... and so on.

But that is not the whole definition. We call them *vector spaces* because the following two operations are defined in these spaces: the vector addition and the scalar multiplication. If we view a vector in $\mathbf{R}^n$ as a column matrix of size $n \times 1$, then the vector addition and scalar multiplication are in fact the same as the matrix addition and scalar multiplication that are restricted to these $n \times 1$ matrices (vectors).

Observation: It seems that $\mathbf{R}$ "lives" in $\mathbf{R}^2$, $\mathbf{R}^2$ "lives" in $\mathbf{R}^3$, $\mathbf{R}^3$ "lives" in $\mathbf{R}^4$, ... and so on.

$\mathbb{R}^3$

$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$



So a vector space can contain a "smaller" vector space inside it (which we shall use the term *subspace* for it), but does a subspace have to be the special examples shown in the above pictures? The answer is no but we need a better understanding on how a vector space behaves with respect to the two operations defined. A very careful study of this finds that the following ten things are true in a vector space with respect to the addition and scalar multiplication. They are independent from each other (that is, one cannot derive one from the other) but can be directly verified for $\mathbf{R}^n$. This is Theorem 4.1.1 in the book.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and let $a, b \in \mathbb{R}$. W

W $\quad \vec{u}\ ,\ \vec{v} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

1. Closure under vector addition: $\mathbf{u} + \mathbf{v} \in \cancel{\mathbb{R}^n}\ W$
2. Closure under scalar multiplication: $a\mathbf{u} \in \mathbb{R}^n\ W$
3. Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4. Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
5. Additive identity: A vector $\mathbf{0} \in \cancel{\mathbb{R}^n}\ W$ exists such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^n$
6. Additive inverse: A unique additive inverse $-\mathbf{u} \in \mathbb{R}^n$ exists such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in \mathbb{R}^n$
7. Associativity of scalar multiplication: $a(b\mathbf{u}) = (ab)\mathbf{u}$ ✓
8. 1 is the scalar identity: $1\mathbf{u} = \mathbf{u}$ ✓
9. Distributivity of scalars under vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ ✓
10. Distributivity of vectors under scalar addition: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ ✓

Definition. Let W be a non-empty subset of $\mathbb{R}^n$. We say that W is a subspace of $\mathbf{R}^n$ if conditions (1) to (10) in Theorem 4.1.1 also hold for vectors in W.

$$W = \left\{ \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{u}\ 2\times 1},\ \underbrace{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{\vec{v}\ 2\times 1} \right\} \qquad \vec{u} + \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \notin W$$

By this definition, verifying whether a non-empty subset of $\mathbb{R}^n$ is a subspace would involve 10 steps. Luckily this task can be simplified to only three steps since most of the 10 things are always true for any subset of $\mathbb{R}^n$. This is stated in Theorem 4.1.2.

Let W be a subset of $\mathbb{R}^n$. Then W is a subspace if and only if the conditions below are satisfied.

i. Zero vector: The zero vector is in W.
ii. Closure under addition: If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$.
iii. Closure under multiplication: If $\mathbf{u} \in W$ and $a \in \mathbb{R}$, then $a\mathbf{u} \in W$.

Example 1. W is the subset of $\mathbf{R}^2$ containing vectors of the form $\begin{bmatrix} x \\ 1 \end{bmatrix}$.

$x \in \mathbb{R}$.

W is not a subspace because

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin W$

Example 2. W is the subset of $\mathbf{R}^3$ containing only the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Yes. Trivial subspace.

Example 3. W is the subset of $\mathbf{R}^n$ containing finitely many vectors including at least one nonzero vector.

$\vec{v} \neq \vec{0}$

$1 \cdot \vec{v}$

$2 \cdot \vec{v}$

$3 \cdot \vec{v}$

$\vdots$

Quiz Question 1. Let W be the subset of $\mathbf{R}^3$ containing vectors of the form $\begin{bmatrix} x \\ x+1 \\ x+2 \end{bmatrix}$, where x is any real numbers, then which of the following statement is correct?

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

A. W is closed under addition;

B. W is closed under scalar multiplication;

C. W is a subspace;

D. W is not a subspace.

Example 4. W is the subset of $\mathbf{R}^2$ containing vectors of the form $\begin{bmatrix} s \\ -2s \end{bmatrix}$ where s is any real number.

$$s\begin{bmatrix}1\\-2\end{bmatrix}$$

$$\begin{bmatrix}s\\-2s\end{bmatrix}+\begin{bmatrix}t\\-2t\end{bmatrix}=\begin{bmatrix}(s+t)\\-2(s+t)\end{bmatrix}\in W$$

$$\begin{bmatrix}1\\-2\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}-1\\-2(-1)\end{bmatrix}\in W$$

$$a\begin{bmatrix}s\\-2s\end{bmatrix}=\begin{bmatrix}as\\-2(as)\end{bmatrix}\in W$$

Example 5. W is the subset of $\mathbf{R}^3$ containing vectors of the form $x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}-2\\3\\1\end{bmatrix}$ where x and y are any real numbers.

$$s\left(x_2\begin{bmatrix}1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\1\end{bmatrix}\right.$$

$$x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}-2\\3\\1\end{bmatrix}+z\begin{bmatrix}1\\1\\0\end{bmatrix}+u\begin{bmatrix}-2\\3\\1\end{bmatrix}+\begin{bmatrix}1\\0\\2\end{bmatrix}\Big)$$

$$=(x+z)\begin{bmatrix}1\\1\\0\end{bmatrix}+(y+u)\begin{bmatrix}-2\\3\\1\end{bmatrix}\in W$$

$$a\left(x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}-2\\3\\1\end{bmatrix}\right)=ax\begin{bmatrix}1\\1\\0\end{bmatrix}+ay\begin{bmatrix}-2\\3\\1\end{bmatrix}\in W$$

Example 6. W is the subset of $\mathbf{R}^3$ containing vectors of the form $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$ where x, y, z are any real numbers, and $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are three given vectors in $\mathbf{R}^3$. Can you generalize this example?

$$W=\{x\vec{V_1}+y\vec{V_2}+z\vec{V_3} : x, y, z \in \mathbb{R}\}$$

Quiz Question 2. Let W be the subset of $\mathbf{R}^3$ containing vectors of the form $\begin{bmatrix} -3x \\ x \\ 5x \end{bmatrix}$, where x is any real numbers, then which of the following statement is NOT correct?

$x \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$

A. W is closed under addition;

B. W is closed under scalar multiplication;

C. W is a subspace;

D. W is not a subspace.

Definition of linear combination. Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be any k vectors of $\mathbf{R}^n$, then for any scalars a_1, a_2, ..., a_k, the vector $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ is called a *linear combination* of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$.

Observations from the examples: Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ be any k vectors of $\mathbf{R}^n$, then if a subset W of $\mathbf{R}^n$ (for any $n \geq 1$) contains all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$, then W is closed under addition and scalar multiplication, hence is a subspace of $\mathbf{R}^n$. In fact, the following statement is also true: if W is not a set that contains all linear combinations of some vectors, then it will not be a subspace.

$$\{ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$$

$$W = \{ a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_k \vec{v}_k \}$$ closed ?

$$+ b_1 \vec{v}_1 + b_2 \vec{v}_2 + \cdots + b_k \vec{v}_k$$

$$(a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \cdots + (a_k+b_k)\vec{v}_k \in W$$

Quiz Question 3. Let W be the subset of $\mathbf{R}^4$ containing vectors of the form $a\begin{bmatrix}-3\\5\\1\\-2\end{bmatrix}+b\begin{bmatrix}0\\0\\4\\1\end{bmatrix}$, where a and b are any real numbers, then which of the following statement is NOT correct?

A. W is closed under addition;

B. W is closed under scalar multiplication;

C. W is a subspace;

D. W is not a subspace.

Example 7: Is the vector $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -4 \\ -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$?

$$a_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + a_3 \begin{bmatrix} -4 \\ -5 \\ 0 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 0 & -4 & 1 \\ -3 & 1 & -5 & 1 \\ 1 & 3 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}_{\vec{b}}$$

Quiz Question 4. To determine whether the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which of the following equations do we need to solve?

A. $\begin{cases} x_1 + 2x_2 & = & 1 \\ 2x_1 - x_2 & = & -1 \end{cases}$

B. $\begin{cases} x_1 - 3x_2 + x_3 & = & 3 \\ x_1 + 2x_2 - x_3 & = & -2 \end{cases}$

C. $\begin{cases} x_1 + 3x_2 + x_3 & = & -3 \\ x_1 - 2x_2 - x_3 & = & 2 \end{cases}$

D. $\begin{cases} x_1 - 3x_2 & = & 3 \\ x_1 + 2x_2 & = & -2 \end{cases}$