Lecture Notes for 11/14/2023

$$\begin{bmatrix} 7 & 2 & 8 \\ -8 & -3 & -8 \\ -2 & -2 & -3 \end{bmatrix}\begin{bmatrix} 7 & 2 & 8 \\ -8 & -3 & -8 \\ -2 & -2 & -3 \end{bmatrix}$$

7.1 Eigenvalues and eigenvectors

7.2 Eigen spaces

A^5

$$7\cdot 7 + 2\cdot(-8) + 8\cdot(-2) = \begin{bmatrix} 17 & * & * \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$
$$49 - 16 - 16$$

A motivational question: given $A = \begin{bmatrix} 7 & 2 & 8 \\ -8 & -3 & -8 \\ -2 & -2 & -3 \end{bmatrix}$, and let n be any positive integer, find a formula for A^n.

Notice the same question for a diagonal matrix would be much easier. For example, if $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then we see that $D^n = \begin{bmatrix} (-3)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 5^3 \end{bmatrix}$.

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

For example, $D^3 = \begin{bmatrix} -27 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 125 \end{bmatrix}$ and $D^4 = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 625 \end{bmatrix}$.

Definition of and eigenvector Let A be a square matrix of size $n \times n$.

• A **non-zero** vector $\mathbf{x} \in \mathbb{R}^n$ is said to be an *eigenvector* of A if there exists a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$. The corresponding scalar λ is called an *eigenvalue* of A (with respect to the eigenvector $\mathbf{x}$). We also say that $(\lambda, \mathbf{x})$ is an *eigenpair*.

$$Ax = \lambda x \qquad \boxed{A_{n\times n}\, x_{n\times 1} = \lambda x}$$

Examples. Verifying whether a given vector is an eigenvector.

$A = \begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. $\quad (\lambda, x) \quad x \neq \vec{0}$

$$Ax = \begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 42 \end{bmatrix} = 14\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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$2\cdot 1 + 4\cdot 3$

$3\cdot 1 + 13\cdot 3 = 42$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ eigenvector with 14 as its corresponding eigenvalue.

$$\left(14, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

$$x = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{NO.}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{NO}$$

Quiz Question 1. Determine which of the following vectors is an eigenvector of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

A. $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$; B. $\mathbf{x} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$; C. $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; D. $\mathbf{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

Does a square matrix always have eigenvalues and eigenvectors? If so how do we find them?

If $(\lambda, \mathbf{x})$ is an eigenpair of the matrix A, then $A\mathbf{x} = \lambda\mathbf{x}$. We can rewrite $\mathbf{x}$ as $I_n\mathbf{x}$ so that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be rewritten as

$x \neq 0$

$Ax - \lambda x = \vec{0}$

$$A\mathbf{x} - \lambda I_n\mathbf{x} = (A - \lambda I_n)\mathbf{x} = \mathbf{0},$$

$(\boxed{A - \lambda I_n})x = 0$

which is a homogeneous linear equation system. Since $\mathbf{x}$ is an eigenvector, it is not $\mathbf{0}$, this means that the equation system has non-trivial solution, which in turn means the matrix $A - \lambda I_n$ must be singular (not invertible), hence ~~it must have zero~~ determinant. So an eigenvalue λ must satisfy the equation $\boxed{\det(A - \lambda I_n) = 0.}$ This is how we find the eigenvalues.

$\lambda x = \boxed{\lambda I} x$

$Bx = 0$

$(A - \lambda I)x = 0$

$\lambda = 1$

Examples. Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$(A + 13)c$

$(A - \lambda I)x = 0$

$\lambda = 1$ $\det(A - \lambda I) = \left|\begin{bmatrix} 2-\lambda & 4 \\ 3 & 13-\lambda \end{bmatrix}\right| = (2-\lambda)(13-\lambda) - 12$

$\begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= 26 - 2\lambda - 13\lambda + \cancel{\lambda^2} - 12 = \lambda^2 - 15\lambda + 14 = 0$

$(\lambda - 1)(\lambda - 14) = 0 \rightarrow \lambda = 1, \quad \lambda = 14$

$\begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix} \sim$ eigenvector?

$-3R_1 + R_2 \to R_2$

$$A = \begin{bmatrix} -3 & 4 & 2 & 0 \\ 0 & 3 & 1 & -5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$|\det(A - \lambda I)| = 0$

$\begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ $x_1 + 4x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\left(1, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right)$ $\left(1, \begin{bmatrix} -8 \\ 2 \end{bmatrix}\right)$

$\ldots \quad \lambda = -3, 3, 0, 2$

$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 4 & 2 & 0 \\ 0 & 3-\lambda & 1 & -5 \\ 0 & 0 & 0-\lambda & 4 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (-3-\lambda)(3-\lambda)(-\lambda)(2-\lambda) = 0$

$-3-\lambda = 0$ $\quad 3-\lambda = 0$

Quiz Question 2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 7 & 0 & 0 \\ -3 & 4 & 0 \\ 2 & 9 & -5 \end{bmatrix}$$

A. $\{7, 0, 0\}$; B. $\{7, 4, -5\}$; C. $\{7, -3, 2\}$; D. $\{2, 4, 0\}$.

We have observed that when we compute $\det(A - \lambda I_n)$, the result is a polynomial of λ with real number coefficients and with degree n. This is called the *characteristic polynomial* of the matrix A. The equation $\det(A - \lambda I_n) = 0$ is called the characteristic equation, the solution set of the characteristic equation (namely the set of all eigenvalues of A) is called the *spectrum* of A.

Let $f(x)$ be a polynomial of degree n and with real number coefficients. A fundamental theorem in algebra states that the equation $f(x) = 0$ has exactly n solutions if we accept complex numbers as our solutions and also count a solution of multiplicity k as k solutions.

Example. If A has size 8×8 and $\det(A - \lambda I_8) = \lambda^2(3-\lambda)(-4-\lambda)^3(7-\lambda)(-10-\lambda)$, find the eigenvalues of A (with their multiplicities).

$$\begin{bmatrix} 1 & & & * \\ 0 & 2 & & \\ 0 & 0 & 2 & \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Example. Find the eigenvalues of $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix}$$

$$(1-\lambda)(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$= \lambda^2 + 16 = 0 \quad \lambda^2 = -16 \quad \lambda = \pm 4i$$

Example. Find the eigenvalues of $A = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 & 0 & 0 \\ 8 & -2 & 4 & 0 & 0 & 0 \\ -3 & 1 & 0 & 4 & 0 & 0 \\ 0 & 6 & -2 & 1 & -3 & 0 \\ -1 & 2 & 9 & 0 & -4 & -3 \end{bmatrix}$

$\lambda = 7, \quad m = 1$

$\lambda = 4, \quad m = 3$

$\lambda = -3, \quad m = 2$

Quiz Question 3. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -5 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

A. $(2-\lambda)(7-5\lambda+\lambda^2)$; B. $(2-\lambda)(1-5\lambda+\lambda^2)$;

C. $(2-\lambda)^3$; D. $(2-\lambda)(5+7\lambda+\lambda^2)$.

$b^2 - 4ac$

$= 25 - 4 \cdot 1 \cdot 7$

$= 25 - 28$

$= -3$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -5 & 3-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(3-\lambda)(2-\lambda)+1\right]$$

$$6 - 5\lambda + \lambda^2 + 1$$

• **Eigenspace, algebraic multiplicity and geometric multiplicity.**

$(\lambda-2)^3(\quad)(\quad)=0$

$\lambda=2$

Let λ_0 be a given eigenvalue of a matrix A. The *eigenspace* of A with respect to λ_0 is the null space of the matrix $(A-\lambda_0 I_n)$ (which is the set that contains all eigenvectors of A with respect to λ_0 together with the zero vector). This is denoted by the notation $E_{\lambda_0}(A)$ in the book.

Also, the nullity of $(A-\lambda_0 I_n)$, namely the dimension of the eigenspace $E_{\lambda_0}(A)$, is called the *geometric multiplicity* of λ_0. Since we know $\det(A-\lambda_0 I_n)=0$, the nullity of $(A-\lambda_0 I_n)$ is at least one. That is, the geometric multiplicity of an eigenvalue is always greater than or equal to one.

$(A-\lambda I)x=0$

• The geometric multiplicity of an eigenvalue can never be greater than the algebraic multiplicity of the eigenvalue.

Examples. Find the eigenvalues and determine their geometric and algebraic multiplicities for the given matrices.

$$A=\begin{bmatrix}2 & -1 & 4 & 3\\ 0 & 3 & 2 & -5\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 5\end{bmatrix}$$

2, 3, 1, 5

a.m. = 1

g.m. = 1

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$A - 3I$

$$\begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -5 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1, a.m. = g.m. = 1

−2 " "

3, a.m. = 2

g.m = 1

Quiz Question 4. Given that $\lambda = 4$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 9 & -3 \\ 1 & 5 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -3 \\ 1 & 5 & -4 \end{bmatrix}$$

find its geometric multiplicity.

A. 0; B. 1; C. 2; D. 3

A for credit