Lecture Notes for 11/16/2023

7.1 Eigenvalues and eigenvectors

$Ax = \lambda x \quad x \neq \vec{0}$

7.2 Eigenspaces

7.3 Similarity and diagonalization

Procedure for finding the eigenvalues and eigenvectors of a matrix:

1. Compute the determinant $\boxed{\det(A-\lambda I)}$ to obtain the characteristic polynomial; of A

2. Solve the characteristic equation $\det(A - \lambda I) = 0$ to obtain the eigenvalues;

3. For each eigenvalue λ obtained, solve the linear equation system $\det(A - \lambda I)\mathbf{x} = \mathbf{0}$ to obtain the eigenvectors (the set of all eigenvectors is the null space of the matrix $A - \lambda I$).

$(A - \lambda I)x = 0$

Example.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix},$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 4 \\ 3 & 13-\lambda \end{bmatrix} \\ &= (2-\lambda)(13-\lambda) - 12 \\ &= \lambda^2 - 15\lambda + 14 \\ &= (\lambda - 1)(\lambda - 14) = 0 \end{aligned}$$

So the eigenvalues are 1 and 14. To find the eigenspace corresponding to $\lambda = 1$: a.m. = 1, g.m. = 1

a.m. = 2 $\quad A - 1 \cdot I \quad \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \quad x_1 = -4x_2, \; x_2 = x_2$

$$(A - \lambda I)\mathbf{x} = \begin{bmatrix} 2-1 & 4 \\ 3 & 13-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

This equation system reduces to just one equation $x_1 + 4x_2 = 0$ with x_2 being the free variable. So the eigen space is $\{x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} : x_2 \in \mathbb{R}\}$.

In particular, $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 1$.

$x_2 = 2 \quad \begin{bmatrix} -8 \\ 2 \end{bmatrix}$

Similarly, the eigen space corresponding to $\lambda = 14$ is obtained by solving the equation system

$$(A - 14I)\mathbf{x} = \begin{bmatrix} 2-14 & 4 \\ 3 & 13-14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -12 & 4 \\ 3 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 \\ -12 & 4 \end{bmatrix} \xrightarrow{4R_1+R_2 \to R_2} \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{3} x_2$

$x_2 = x_2$

x_2 is the free variable so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}.$$

$\operatorname{Span}\left(\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right)$

Since x_2 is the free variable, we can choose it to be any nonzero value. By choosing it to be 3, we get a particular eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. (The choice is so that the eivenvector does not contain fraction entries.)

Quiz Question 1. Given that $\lambda = 3$ is an eigenvalue of the matrix $\begin{bmatrix} 5 & 1 & 1 \\ -1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, in order to find its corresponding eigenspace, which of the following equations we have to solve?

A. $\begin{bmatrix} 5 & 1 & 1 \\ -1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; B. $\begin{bmatrix} 8 & 1 & 1 \\ -1 & 5 & 3 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$;

C. $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; D. $\begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Returning to the example at beginning where $A = \begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}$ has eigenpairs $\left(1, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right)$ and $\left(14, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$. Let us consider the matrix $M = \begin{bmatrix} -4 & 1 \\ 1 & 3 \end{bmatrix}$, we have

$\det(M) = -13$

$$M^{-1} = -\frac{1}{13}\begin{bmatrix} 3 & -1 \\ -1 & -4 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -3 & 1 \\ 1 & 4 \end{bmatrix}$$

Let us now compute $M^{-1}AM$ and see what we get.

$$\begin{aligned} M^{-1}AM &= \frac{1}{13}\begin{bmatrix} -3 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 13 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 3 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} -3 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4 & 14 \\ 1 & 42 \end{bmatrix} \\ &= \frac{1}{13}\begin{bmatrix} 13 & 0 \\ 0 & 182 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 14 \end{bmatrix} \end{aligned}$$

$3 + 39$

$-42 + 42 = 0$

$M\left(M^{-1} A^n M = \begin{bmatrix} 1 & 0 \\ 0 & 14^n \end{bmatrix}\right) M^{-1}$

$A^n = M \begin{bmatrix} 1 & 0 \\ 0 & 14^n \end{bmatrix} M^{-1}$

$M^{-1}AM$

$M^{-1}AM \cdot M^{-1}AM = \begin{bmatrix} 1 & 0 \\ 0 & 14^2 \end{bmatrix}$

$M^{-1}A^2M\, M^{-1}AM = \begin{bmatrix} 1 & 0 \\ 0 & 14^3 \end{bmatrix}$

$M^{-1}A^3M$

Two square matrices A and B of the same size $n \times n$ are said to be *similar* to each other, if there exists an invertible matrix M such that $B = M^{-1}AM$. If A is similar to a diagonal matrix D, then A is said to be *diagonalizable*. In the example above, A is diagonalizable and is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 14 \end{bmatrix}$. Do you remember the motivational question at the beginning of the chapter that I was talking about?

Suppose for the matrix A in the above example, we need to find a general formula for A^n for any integer $n \geq 1$.

$$B = M^{-1} A M \qquad |B| = |M^{-1} A M| = |M^{-1}| \cdot |A| \cdot |M|$$

$$|M^{-1}| = \frac{1}{|M|} \qquad = |A|$$

If A and B are similar to each other, then they share some common properties such as (Theorem 7.3.1):

• $\det(A) = \det(B)$ (which implies that A is invertible if and only if B is invertible);

• A and B have the same spectrum;

• A and B have the same eigenspaces; ✓

• For any integer $n \geq 1$, A^n and B^n are also similar to each other hence also have the same spectrum and the same eigenspaces.

A is diagonalizable if and only the algebraic multiplicity and geometric multiplicity are equal for every eigenvalue of A. The columns of M consist of the vectors from the basis of each eigen space. Alternatively, this condition can be stated as: the sum of the geometric multiplicities of the eigenvalues equals n. Or, the sum of the dimensions of the eigenspaces of A is equal to n.

Example. The matrix

$$A = \frac{1}{3}\begin{bmatrix} -9 & 0 & 0 \\ 16 & -1 & -16 \\ -16 & -8 & 7 \end{bmatrix}$$

$x_2 \begin{bmatrix} \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \end{bmatrix}$

has spectrum $\{-3, 5\}$, the algebraic multiplicity and geometric multiplicity of -3 are both 2, the algebraic multiplicity and geometric multiplicity of 5 is both 1. A basis for the eigen space of -3 is -3: $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$, a basis for the eigen space of 5 is $\overset{5}{\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}}$. If we choose $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, then we will have

$M^{-1}AM$

$$\begin{aligned} M^{-1}AM &= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 1 \\ -2 & -1 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -9 & 0 & 0 \\ 16 & -1 & -16 \\ -16 & -8 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}. \end{aligned}$$

$= \begin{bmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{bmatrix}$

The details of this calculation are left for you to verify.

Question: what if we choose $M = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \\ \begin{matrix} -3 & 5 & -3 \end{matrix} \end{array}$?

$M^{-1}AM = \begin{bmatrix} -3 & & \\ & 5 & \\ & & -3 \end{bmatrix}$

$B = M^{-1} A M$

Quiz Question 2. If A is a 4×4 matrix, then in which of the following cases A CANNOT be diagonalized?

A. A has spectrum $\{4, 7\}$, the algebraic multiplicities of 4 and 7 are both 2, while the geometric multiplicity of 4 is 2 and the geometric multiplicity of 7 is 1.

B. A has spectrum $\{1, 2, -1, 4\}$;

C. A has spectrum $\{0, 3, -2, -1\}$;

D. A has spectrum $\{1, 3\}$ and the geometric multiplicity of 1 is 3.

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Quiz Question 3. Given that A has size 3×3 and has eigen pairs $\left(4, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$, $\left(-2, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right)$ and $\left(1, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right)$. If we choose

$$M = \begin{array}{c} \begin{bmatrix} 0 & 1 & 2 \\ -2 & 0 & 3 \\ 1 & -1 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow \; & \uparrow & \; \uparrow \\ 1 \; & 4 & \; -2 \end{matrix} \end{array},$$

then which of the following is $M^{-1}AM$?

A. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ D. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Cases when we can determine quickly whether a matrix A can be diagonalized.

• If A is of size $n \times n$ and has n distinct eigenvalues (since in this case the geometric multiplicity of each eigenvalue is one and they sum to n);

• If A is a triangular matrix: in this case the eigenvalues of A are the entries on the diagonal line of A. For each eigenvalue λ that has algebraic multiplicity 1 (meaning it only appears on the diagonal line once), we know it has geometric multiplicity 1 as well so there is no need to check on such an eigenvalue. If an eigenvalue λ has algebraic multiplicity 2 or more, we need to examine $A - \lambda I_n$. But this is almost in an echelon form so it should be relatively easy to reduce it to an echelon form so we can see what is the number of free variables (which equals the geometric multiplicity of the eigenvalue).

Example. $\begin{bmatrix} 2 & -2 & -1 & 0 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$\underline{\lambda = 2 \quad m = 3} \quad g.m = 1$

$\underline{\lambda = 4 \quad m = 1.}$

$$(A - 2 \cdot I)X = \begin{bmatrix} 0 & -2 & -1 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

what if $A = \begin{bmatrix} 4 & 1 & 1 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\lambda = 4$ $\quad$ $gm = 1$, $a.m. = 1$

$\lambda = 2$ $\quad$ $a.m = 2$, $\underline{g.m = 2}$

$\lambda = 1$ $\quad$ $\underline{a\,m = 1}$, $gm = 1$

$$A - 2I = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\underline{\text{Yes.}}$

Quiz Question 4. Which of the following matrices cannot be diagonalized?

A. $\begin{bmatrix} -3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$ B. $\begin{bmatrix} -3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$(A - \lambda I) x = 0$

C. $\begin{bmatrix} -3 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -7 & 2 & 3 & -6 \end{bmatrix}$

$\begin{bmatrix} -4 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$