# Lecture Notes for 11/2/2023

## 6.1 Linear transformations between Euclidean spaces

• A transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^k$ is similar to a special linear function from $\mathbb{R}$ to $\mathbb{R}$.

Examples of functions from $\mathbb{R}^n$ to $\mathbb{R}^k$.

$f(x) = x^2 - 2x + 1$ for any $x \in \mathbb{R}$;

$f(2) = 2^2 - 2\cdot 2 + 1 = 1$  $f(-3) = (-3)^2 - 2\cdot(-3) + 1$

$\mathbb{R}^1 \longrightarrow \mathbb{R}^1$  $= 9 + 6 + 1 = 16$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $f(x_1, x_2) = 5x_1x_2 + x_1^3 - 4$ for any $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$;

$\mathbb{R}^2 \longrightarrow \mathbb{R}^1$

$\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f(x) = 5x - 1$ for any $x \in \mathbb{R}$;

$f(x+y)$
$= 5(x+y) - 1$
$= 5x + 5y - 1$
$\times \stackrel{?}{=} f(x) + f(y)$
$5x - 1 + 5y - 1$

$f(x) = -8x$ for any $x \in \mathbb{R}$;

However when the domain is no longer $\mathbb{R}$, we change the name from "function" to "transformation" and use the notation $T$ instead of $f$.

$$T(x) = \begin{bmatrix} 3x \\ x^3 \\ -1 \\ 2x+1 \end{bmatrix} \text{ for any } x \in \mathbb{R};$$

$\mathbb{R}^1 \longrightarrow \mathbb{R}^4$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ -x_2 + 4x_3 \end{bmatrix} \text{ for any } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3.$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1^2 \\ x_1 - x_2 \\ 2x_1x_2 \\ \sin(x_1)\cos(x_2) \end{bmatrix}.$$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^4$

In general, for each $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x})$ is a unique vector in $\mathbb{R}^k$. $\mathbb{R}^n$ is called the *domain* of $T$ and $\mathbb{R}^m$ is called the *codomain* of $T$. It's general form is

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_k(x_1, x_2, ..., x_n) \end{bmatrix}$$

$\mathbb{R}^n \to \mathbb{R}^k$

where each of $f_1(x_1, x_2, ..., x_n)$, ..., $f_k(x_1, x_2, ..., x_n)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

As we are in linear algebra, we are only interested in the "linear transformations". Conceptually, this means transformations that will take a vector space (including the subspaces) to a vector space such that the structure of the vector space is preserved.

• $T: \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is called a *linear transformation* if it satisfies the following two conditions:

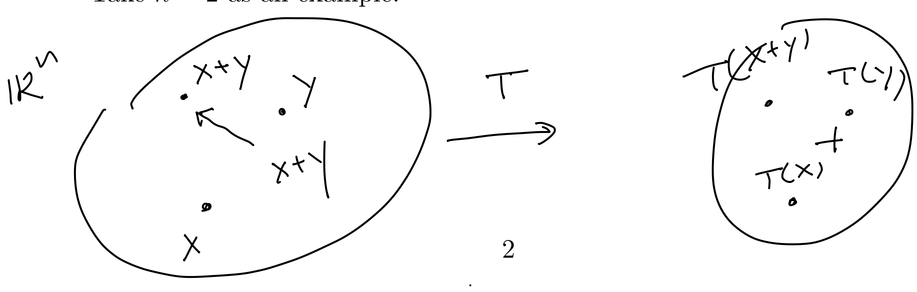
$\mathbb{R}^n \longrightarrow \mathbb{R}^k$

1. For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$;
2. For any scalar $c \in \mathbb{R}$ and any $\mathbf{x} \in \mathbb{R}^n$, $T(c\mathbf{x}) = cT(\mathbf{x})$.

$c\ T(x)$

Take $n = 2$ as an example.

$\mathbb{R}^k$






$f(x) = 8x$

$f(x+y) = 8(x+y)$

$= 8x + 8y = f(x) + f(y)$

Examples.

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 2x_1 + x_2 \end{bmatrix}$

$$T\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 - 2cx_2 \\ 2cx_1 + cx_2 \end{bmatrix}$$

$$= c\underbrace{\begin{bmatrix} x_1 - 2x_2 \\ 2x_1 + x_2 \end{bmatrix}} = cT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ 0 \\ x_1^2 \end{bmatrix}$

No. Example: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix},$

$$2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\underline{T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)} = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \qquad 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right.$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ 0 \\ x_1^2 \end{bmatrix}$

Is there a simple "rule of thumb" that we can use to detect whether a transformation is linear? The answer is yes.

Here is the rule: in order for

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2, ..., x_n) \\ f_2(x_1, x_2, ..., x_n) \\ \vdots \\ f_k(x_1, x_2, ..., x_n) \end{bmatrix}$$

to be a linear transformation, each of $f_1(x_1, x_2, ..., x_n)$, ..., $f_k(x_1, x_2, ..., x_n)$ must be a linear combination of $x_1$, $x_2$, ..., $x_n$.

$2x_1 - x_2 + 4x_3 = 6$

Examples.

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 4x_3 \\ -x_1 + 2x_3 \end{bmatrix}$. $\begin{matrix} = 6 \\ = -2 \end{matrix}$

Notice that in this case $\begin{bmatrix} 2x_1 - x_2 + 4x_3 \\ -x_1 + 2x_3 \end{bmatrix}$ can be expressed in terms of matrix multiplication $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, hence $T$ is a linear transformation with the matrix $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 0 & 2 \end{bmatrix}$, therefore we can write

$$= x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We say that $T$ is a linear transformation with the matrix $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 0 & 2 \end{bmatrix}$.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 + 1 \\ x_1 + x_3 - 2 \end{bmatrix}$$ No

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + x_3 \\ 0 \end{bmatrix}$$ Yes

Quiz Question 1. Which of the following is a linear transformation?

A. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 x_2 x_3$;

B. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 4x_2 - x_3 + 1 \\ -x_1 + x_2 - 3x_3 + 2 \\ 0 \end{bmatrix}$

C. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underline{\begin{bmatrix} -4x_1 + 5x_2 \\ 3x_1 - 2x_2 \\ x_1 + x_2 \end{bmatrix}} = Ax$

D. $T: \mathbb{R} \longrightarrow \mathbb{R}^3$ is defined by $T(x_1) = \begin{bmatrix} 0 \\ 1 \\ 2x_1 \end{bmatrix}$

$\underline{T(x) = Ax}$

$$T(x+y) = T(x) + T(y)$$
$$A(x+y) = Ax + Ay$$

$$T(cx) = A(cx) = cAx = cT(x)$$

$\underline{v_1, v_2, \dots, v_k}$

Why the "rule of thumb" given above works? That is because we have the following theorem. $T(x) = \underline{A}x$

Theorem 6.1.1. If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a linear transformation, then there exists a matrix $A$ of size $k \times n$ such that for any $\mathbf{x} \in \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$. That is, a linear transformation is always in the form of a matrix transformation. In fact, if we let $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$, then the columns of $A$ are exactly $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, ..., $T(\mathbf{e}_k)$.

$\underline{T(e_1)}$

For example, if

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}4\\-2\\3\end{bmatrix}, \ T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\-2\end{bmatrix}, \ T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix},$$

then

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}4 & 1 & 0\\-2 & 1 & 1\\3 & -2 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

If

$T$ is linear $\quad T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\-2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\1\\2\end{bmatrix}$$

then $A = \begin{bmatrix}0 & 5\\1 & 1\\-2 & 2\end{bmatrix}$

Quiz Question 2. If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation and

$$2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}4\\-2\end{bmatrix}, \qquad 3T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\9\end{bmatrix},$$

then the matrix of $T$ is

A. $\begin{bmatrix}4 & -2\\0 & 9\end{bmatrix}$

B. $\begin{bmatrix}2 & 0\\-1 & 9\end{bmatrix}$;

C. $\begin{bmatrix}2 & 0\\-1 & 3\end{bmatrix}$;

D. $\begin{bmatrix}2 & -1\\0 & 3\end{bmatrix}$.

$$2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}4\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$3T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\9\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix}$$

But why Theorem 6.1.1 is true? Here is an explanation.

Given a vector

$$
\begin{aligned}
T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) &= T\left(\begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix}\right) \\
&= T\left(x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \\
&= x_1 T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) + \cdots + x_n T\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \\
&= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \cdots + x_n T(\mathbf{e}_n) \\
&= A\mathbf{x}
\end{aligned}
$$

$A = (T(e_1), \cdots, T(e_n))$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Quiz Question 3. Given that $T$ is a linear transformation and that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix},$$

find $T\left(\begin{bmatrix}3\\-4\end{bmatrix}\right)$.

$$A = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix} \quad T(x) = Ax$$

$$\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix} = \begin{bmatrix}11\end{bmatrix}$$

A. $\begin{bmatrix}3\\-4\end{bmatrix}$; B. $\begin{bmatrix}1\\-1\end{bmatrix}$; C. $\begin{bmatrix}-2\\3\end{bmatrix}$; D. $\begin{bmatrix}11\\-15\end{bmatrix}$.

$$\mathbb{R}^n \longrightarrow \mathbb{R}^k,$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix} \; ?$$

$$A = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

We can find the matrix $A$ of $T$ if the explicit formula for $T$ is written out, or if we know $T(\mathbf{e}_1)$, ..., $T(\mathbf{e}_n)$ for the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$. What if we only know $T(\mathbf{b}_1)$, ..., $T(\mathbf{b}_n)$ for a non-standard basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$?

Example. If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation and $T\left(\begin{bmatrix}-3\\4\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$, $T\left(\begin{bmatrix}-4\\5\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix}$, that is the matrix of $T$?

$B = \left\{\begin{bmatrix}-3\\4\end{bmatrix}, \begin{bmatrix}-4\\5\end{bmatrix}\right\}$ $C = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\right\}$

We want to know $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ since $A = (T(\mathbf{e}_1), T(\mathbf{e}_2))$. But we only know $T\left(\begin{bmatrix}-3\\4\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}-4\\5\end{bmatrix}\right)$ in this example. However, $\mathcal{B} = \{\begin{bmatrix}-3\\4\end{bmatrix}, \begin{bmatrix}-4\\5\end{bmatrix}\}$ is a basis for $\mathbb{R}^2$, so $\mathbf{e}_1$ and $\mathbf{e}_2$ are linear combinations of $\mathbf{b}_1, \mathbf{b}_2$. So how do we find the combination coefficients? Recall these coefficients are exactly the coordinate vectors of $\mathbf{e}_1$ and $\mathbf{e}_2$ under the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. So

$A = CM_B^{-1}$

$T(e_1), T(e_2)$

$$[\mathbf{e}_1]_{\mathcal{B}} = \underbrace{\begin{bmatrix}-3 & -4\\4 & 5\end{bmatrix}^{-1}}_{M_B^{-1}} \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5 & 4\\-4 & -3\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5\\-4\end{bmatrix}$$

$\begin{bmatrix}1\\0\end{bmatrix}$

$C = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$

This means that $\mathbf{e}_1 = 5\begin{bmatrix}-3\\4\end{bmatrix} - 4\begin{bmatrix}-4\\5\end{bmatrix}$, so

$$T(\mathbf{e}_1) = 5T(\begin{bmatrix}-3\\4\end{bmatrix}) - 4T(\begin{bmatrix}-4\\5\end{bmatrix}) = 5\begin{bmatrix}1\\2\end{bmatrix} - 4\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}5\\-4\end{bmatrix}$$

Similarly,

$= \begin{bmatrix}-3\\14\end{bmatrix}$

$$[\mathbf{e}_2]_{\mathcal{B}} = \begin{bmatrix}5 & 4\\-4 & -3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}4\\-3\end{bmatrix}.$$

Thus, $A = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}5 & 4\\-4 & -3\end{bmatrix} = \begin{bmatrix}-3 & -2\\14 & 11\end{bmatrix}$.

$C \cdot M_B^{-1}$

In general, if we let $C$ be the matrix with columns $T(\mathbf{b}_1)$, ..., $T(\mathbf{b}_n)$ and $M_{\mathcal{B}}$ be the matrix with columns $\mathbf{b}_1$, ..., $\mathbf{b}_n$, then the matrix $A$ of $T$ is simply $CM_{\mathcal{B}}^{-1}$.

Quiz Question 4. Given that $\mathcal{B} = \{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}\}$ is a basis for $\mathbb{R}^2$ and $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \underline{\begin{bmatrix}1\\0\end{bmatrix}}, \quad T\left(\begin{bmatrix}4\\3\end{bmatrix}\right) = \underline{\begin{bmatrix}-2\\1\end{bmatrix}}.$$

$M_B = \begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}$

$\det(M_B) = 1$

Find the matrix $A$ of $T$, that is, $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = A\begin{bmatrix}x_1\\x_2\end{bmatrix}$.

$\underline{M_B^{-1}} = \begin{bmatrix}3 & -4\\-2 & 3\end{bmatrix}$

A. $\begin{bmatrix}3 & -10\\-2 & 7\end{bmatrix}$; B. $\begin{bmatrix}7 & -10\\-2 & 3\end{bmatrix}$; C. $\begin{bmatrix}-1 & -2\\2 & 3\end{bmatrix}$; D. $\begin{bmatrix}3 & -2\\2 & -1\end{bmatrix}$.

$C = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$