Lecture Notes for 11/21/2023

8.1 Introduction to vectors

8.2 Vector operations

In this chapter we concentrate on the applications of vectors in the 2 and 3 dimensions, as they are used in physics.

Key concepts:

• **vector**: a line segment with a direction represented by an arrow.

• **initial point and terminal point** of a vector: the points at which a vector starts and ends respectively (the arrow tip is at the terminal point).

$\vec{v}$

• Common notations for a vector: lowercase boldfaced letter such as $\mathbf{v}$, an arrow over a lowercase letter such as $\vec{v}$, a directed line segment such as $\overrightarrow{AB}$ where A is the initial point and B is the terminal point.

• **position vector**: a vector with the origin as the initial point. Any vector in standard position can be uniquely identified by the terminal point hence can be expressed using the coordinates (called *components* in this book) of the terminal point. Three notations are used. For example, the standard position vector ending at point $(2, -3, 5)$ can be expressed using the square bracket notation $[2, -3, 5]$, the angle bracket notation $\langle 2, -3, 5\rangle$, or the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ notation $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ stand for the vectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$ and $\langle 0, 0, 1\rangle$ respectively.

• **zero vector**: a vector whose components are all zero.

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$

$\begin{bmatrix}2\\-1\\3\end{bmatrix}$ $\quad [2, -1, 3]$, $\langle 2, -1, 3\rangle$

$2\vec{i} - \vec{j} + 3\vec{k}$

Examples.

$3, -1, -1$

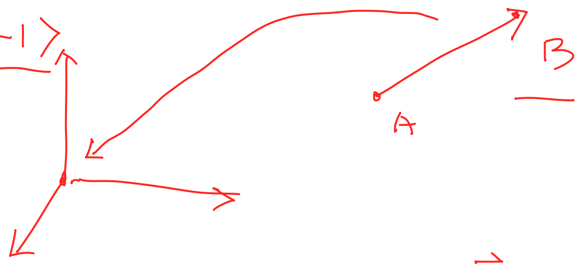
A

Given that the initial and terminal points of $\mathbf{w}$ are $(-3, 1, 1)$ and $(1, -5, 0)$ respectively. Find the position vector of $\mathbf{w}$.

$= B$

$\vec{v} = \langle 4, -6, -1 \rangle$

$4\vec{i} - 6\vec{j} - \vec{k}$



$\langle 3, 1, 5 \rangle$

$-3 \quad +1 \quad -5$

$\langle 0, 0, 0 \rangle$

$\vec{u}$

Given that the position vector of $\mathbf{u}$ is $\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ and the initial point of $\mathbf{u}$ is $(2, 7, -9)$, find the terminal point of the vector.

A $\qquad$ B

$\vec{u} = \langle B - A \rangle$

$(1, 2, -3) = B - (2, 7, -9)$

$B = (1, 2, -3) + (2, 7, -9) = (3, 9, -12)$

Given that the position vector of $\mathbf{v}$ is $\langle 1, 0, 3 \rangle$ and the terminal point of $\mathbf{v}$ is $(0, -2, 4)$, find the initial point of the vector $\mathbf{v}$.

$B \quad + \cancel{A}$ $\qquad$ $+A$

$(1, 0, 3) = (0, -2, 4) - A$

$A = (0, -2, 4) - (1, 0, 3)$

$= (-1, -2, 1)$

Quiz Question 1. If $\overrightarrow{AB} = [1, -3, 4]$ and $B = (3, -2, -1)$, find the initial point A.

A. $A = (2, 1, -5)$; B. $A = [2, 1, -5]$; C. $A = (4, -5, 3)$; D. $A = [4, -5, 3]$.

$$B - A = (1, -3, 4)$$

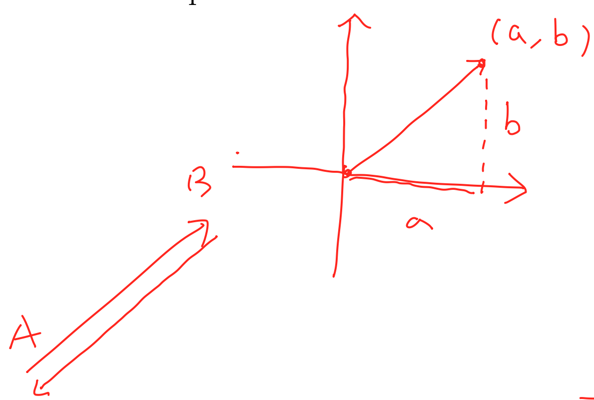
$$+A \qquad\qquad +A$$

$$B = (1, -3, -4) + A$$

$$A = \cancel{(1, -3, 4} \quad (3, -2, -1) - (1, -3, -4)$$

The **magnitude** of a vector: If $\mathbf{v} = [a, b] = a\mathbf{i} + b\mathbf{j}$, then $||\mathbf{v}|| = \sqrt{a^2 + b^2}$. Similarly, if $\mathbf{v} = \langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $||\mathbf{v}|| = \sqrt{a^2 + b^2 + c^2}$.

Examples.



A B

$(x_1, y_1, z_1), \ (x_2, y_2, z_2)$

$\downarrow$

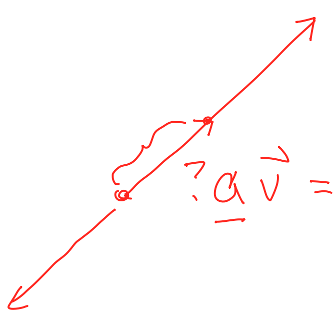
$[x_2 - x_1, y_2 - y_1, z_2 - z_1]$

$|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$\vec{v} = [1, -1, 4]$, $||\vec{v}|| = \sqrt{1+1+16} = \sqrt{18} = \boxed{3\sqrt{2}}$

Unit vector: a vector whose magnitude is 1.

Question: given a vector $\mathbf{v} \neq \mathbf{0}$, how to find a unit vector that has the same direction as $\mathbf{v}$?



$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{3\sqrt{2}}\vec{v} = \left[\frac{1}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right]$

$?\ a\vec{v} = [a, -a, 4a]$ $\quad |a\vec{v}| = \sqrt{a^2 + a^2 + 16a^2}$

$= \sqrt{18a^2} = \sqrt{18}\cdot\sqrt{a^2} = a\sqrt{18} = a\,||v||$

Quiz Question 2. Given that $\mathbf{v} = [-2, 3, 5]$, find the unit vector with direction **opposite** to $\mathbf{v}$.

A. $[-\frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}}]$ B. $[\frac{2}{\sqrt{38}}, -\frac{3}{\sqrt{38}}, -\frac{5}{\sqrt{38}}]$

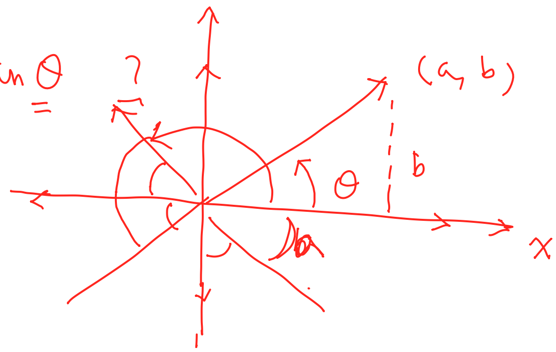
C. $[-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{5}{\sqrt{6}}]$ D. $[\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{5}{\sqrt{6}}]$

$|v| =$

The **direction** (angle) of a vector in $\mathbb{R}^2$:

$\frac{b}{a} = \tan\theta$

$\theta = \tan^{-1}\frac{|b|}{|a|}$



I: $\tan^{-1}(\frac{b}{a})$

II: $180^\circ - \tan^{-1}(\frac{b}{|a|})$

III: $180^\circ + \tan^{-1}(\frac{b}{a})$

IV: $360^\circ - \tan^{-1}(|\frac{b}{a}|)$

Example. The direction of $[-1, 3]$ (it is in II) is: $180^\circ - \tan^{-1}(\frac{3}{1})$

Quiz Question 3. Find the direction of the vector $-3\mathbf{i} - 5\mathbf{j}$.

A. $\tan^{-1}(\frac{3}{5})$; B. $180° + \tan^{-1}(\frac{3}{5})$; $\tan^{-1}(\frac{5}{3})$; D. $180° + \tan^{-1}(\frac{5}{3})$.

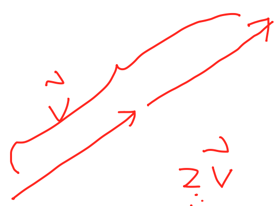
Vector operations and geometric interpretations.

We have already learned the scalar multiplication and addition of the vectors. So we will only discuss the geometric interpretations of these operations.

The geometric interpretation of the scalar multiplication:

If $k > 0$, then $k\mathbf{v}$ has the same direction as $\mathbf{v}$, with the magnitude of $k\mathbf{v}$.

If $k < 0$, then $k\mathbf{v}$ has the opposite direction as $\mathbf{v}$, with the magnitude of $|k|\mathbf{v}$.



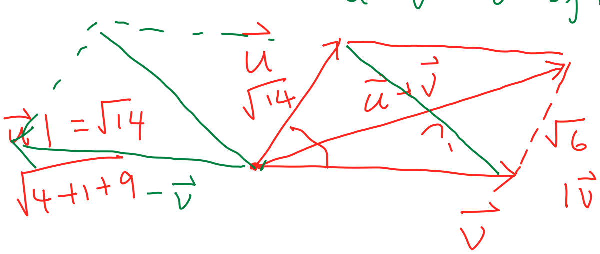
$2\vec{v}$ $3\vec{v}$, $\frac{1}{2}\vec{v}$

The geometric interpretation of addition (and subtraction).

$\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$, $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$, $\vec{u}+\vec{v}$

$\vec{u} - \vec{v} = \vec{i} - 3\vec{j} + 2\vec{k}$

$= 3\vec{i} + \vec{j} + 4\vec{k}$



$|\vec{u}| = \sqrt{14}$

$= \sqrt{4+1+9}$ $-\vec{v}$

$\vec{u}+\vec{v}$?

$|\vec{v}| = \sqrt{1+4+1} = \sqrt{6}$

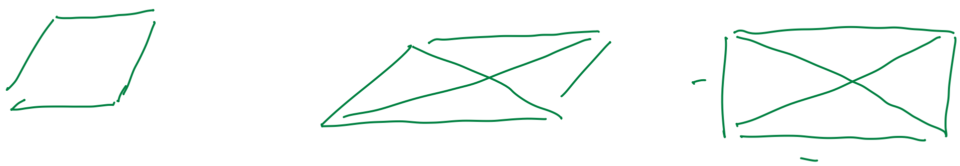
$|\vec{u}+\vec{v}| = \sqrt{9+1+16} = \sqrt{26}$

$\vec{u} + (-\vec{v}) = \vec{u} - \vec{v}$ $|\vec{u}-\vec{v}| = \sqrt{1+9+4} = \sqrt{14}$

Example. If the vectors $\mathbf{u} = [1, -1, 2]$ and $\mathbf{v} = [0, 3, 1]$ are the two sides of a parallelogram, find the lengths of the two diagonals of the parallelogram.

$\vec{u} + \vec{v} = [1, 2, 3]$ $\quad |\vec{u} + \vec{v}| = \sqrt{1+4+9} = \sqrt{14}$

$\vec{u} - \vec{v} = [1, -4, 1]$ $\quad |\vec{u} - \vec{v}| = \sqrt{1+16+1} = \sqrt{18}$ "?



Quiz Question 4. If the vectors $\mathbf{u} = [2, -1]$ and $\mathbf{v} = [2, 4]$ are the two sides of a parallelogram, find the lengths of the two diagonals of the parallelogram.

A. $\sqrt{5}$ and $2\sqrt{5}$ B. $\sqrt{5}$ and $\sqrt{5}$ C. 5 and 5 D. 25 and 25