webwork assignment 7

#15 , #16.

$A_{2\times 2}$

$A_{3\times 3}$: $A^3 + c_1 A^2 + c_2 A + c_3 I = O_{3\times 3}$

$|A - \lambda I| = -\lambda^3 - c_1\lambda^2 - c_2\lambda - c_3$

$| \quad | = \lambda^3 + 2\lambda^2 - 3\lambda + 4$ -

$A^3 + 2A^2 - 3A + 4I = O_{3\times 3}$

$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ $\qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)$

$= \lambda^2 - 6\lambda + 8$

$A^2 + c_1 A + c_2 I = O_{2\times 2}$

Lecture Notes for 11/28/2023

8.3 Dot product

8.4 Cross product

Thursday, 11/30 is a Review Session for Test 3 which covers 6.1–6.4, 7.1–7.3, 8.1–8.4.

- Definition of dot product. Formulation using matrix multiplication.

This can be defined for vectors in $\mathbb{R}^n$ for any $n \geq 2$. If we write vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ as column vectors

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} =$$

1x3 3x1

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

then $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. This is as if we are doing the matrix multiplication $\mathbf{x}^T\mathbf{y}$, or $\mathbf{y}^T\mathbf{x}$. If $\mathbf{x}$ and $\mathbf{y}$ are written in a row matrix form, the formula is the same.

Examples.

$\langle 2, -4, 1\rangle \cdot \langle 1, 3, 5\rangle = 2 \cdot 1 + (-4) \cdot 3 + 1 \cdot 5 = -5.$

$(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \cdot (5\mathbf{i} + \mathbf{j} - 2\mathbf{k}) =$ $10 + 3 - 8 = 5$

$\langle 0, 0, 4\rangle \cdot \langle -2, 1, 0\rangle$

$(4\mathbf{k}) \cdot (-2\mathbf{i} + \mathbf{j}) =$ 0

A few immediate observations from the definition of the dot product:

• The dot product of two vectors is a real number, and it can be positive, negative or zero;

• $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;

$\langle 1, 2, -1 \rangle \cdot \langle 3, 4, 1 \rangle$

$\langle 3, 4, 1 \rangle \cdot \langle 1, 2, -1 \rangle$

• $\mathbf{x} \cdot \mathbf{y} = 0$ does not mean that one of the vectors $\mathbf{x}$, $\mathbf{y}$ needs to be the zero vector.

• $\mathbf{x} \cdot (k\mathbf{y}) = (k\mathbf{x}) \cdot \mathbf{y} = k(\mathbf{x} \cdot \mathbf{y})$;

• $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

Example. If $\mathbf{u} \cdot \mathbf{v} = 3$, $\mathbf{u} \cdot \mathbf{w} = -5$, then $\mathbf{u} \cdot (2\mathbf{v} - 3\mathbf{w}) =?$

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$$u \cdot (2v) - u \cdot (3w)$$

$$= 2(u \cdot v) - 3(u \cdot w)$$

$$= 2 \cdot 3 - 3(-5)$$

$$= 6 + 15 = 21$$

Example. If $\langle 2, -4, c \rangle \cdot \langle 2, 1, 5 \rangle = 0$, then $c =?$

||

$$2 \cdot 2 - 4 \cdot 1 + 5c = 0$$

$$5c = 0$$

$$c = 0$$

Quiz Question 1. Given that $\mathbf{u} \cdot \mathbf{v} = 4$, $\mathbf{v} \cdot \mathbf{w} = 3$, find $(3\mathbf{u} - 2\mathbf{w}) \cdot \mathbf{v}$.

A. 7;   B. 6;   C. 18;   D. -6.

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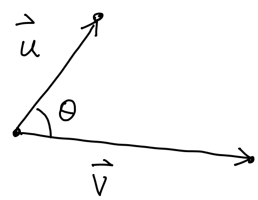
• Geometric meaning of dot product



$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

Example. If $||\mathbf{u}|| = 3$, $||\mathbf{v}|| = 4$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60°, then $\mathbf{u} \cdot \mathbf{v} =$?

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

$$= 3 \cdot 4 \cdot \cos 60^\circ = \frac{12}{2} = 6$$

Example. If $\mathbf{u} = [1, -2, 3]$ and $\mathbf{v} = [0, 2, 1]$, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \cdot \vec{v} = 1 \cdot 0 + (-2) \cdot 2 + 3 \cdot 1 = -4 + 3 = -1$$

$$\|\vec{u}\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\|\vec{v}\| = \sqrt{4+1} = \sqrt{5}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{14} \cdot \sqrt{5}}\right)$$

Example. If $\mathbf{u} = [1, -2, 3]$ and $\mathbf{v} = [1, 2, 1]$, find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \cdot \vec{v} = 1 \cdot 1 + (-2) \cdot 2 + 3 \cdot 1 = 0$$

$$\vec{u} \cdot \vec{v} = \underline{\|\vec{u}\| \cdot \|\vec{v}\|} \cdot \underline{\cos\theta} = 0$$

Theorem. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example. Determine the value of a so that the vectors $\mathbf{u} = [2, 1, -3]$ and $\mathbf{v} = [a, 1, 4]$ are perpendicular to each other.

$\vec{u} \cdot \vec{v} = 2a + 1 - 12 = 0 \quad 2a - 11 = 0$

$2a = 11$

$a = 11/2 = 5.5$

Quiz Question 2. Determine which of the following vectors is perpendicular to the vector $[2, 3, -1]$.

A. $[-2, -3, 1]$ B. $[3, 4, 10]$ C. $[1, 2, 8]$ D. $[1, 0, 1]$

8.4 Cross product

The definition of cross product of two vectors in $\mathbb{R}^3$.

If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{V} \times \vec{u} = \begin{vmatrix} i & j & K \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\vec{u} \times \vec{V}$$

Example. If $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -4 \\ 2 & -1 & -1 \end{vmatrix} = -6i - 7j - 5K$$

$$\begin{vmatrix} i & j & K & i & j \\ 1 & 2 & -4 & 1 & 2 \\ 2 & -1 & -1 & 2 & -1 \end{vmatrix} = -2i - 8j - K - 4K - 4i + j$$

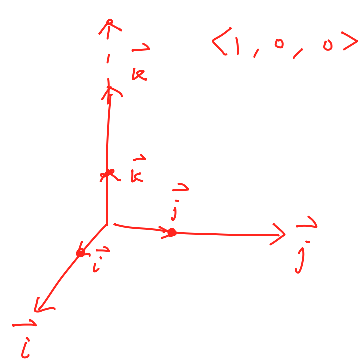
$$= -6i - 7j - 5K$$

$+0 \cdot \vec{k}$

Example. If $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = K \cdot \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= 7K \qquad 3-(-4)$$



Quiz Question 3. If $\mathbf{u} = 3\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$, find $\mathbf{u} \times \mathbf{v}$.

A. $3\mathbf{k}$ B. $3\mathbf{j}$ C. $-3\mathbf{k}$ D. $2\mathbf{i}$

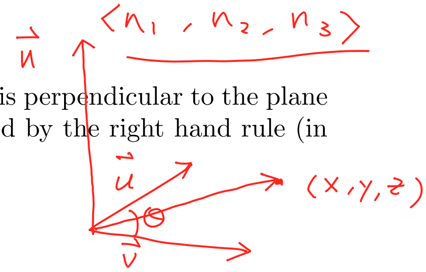
$$\begin{vmatrix} \bar{i} & \bar{j} & k \\ 0 & 3 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

Properties and geometric interpretation of cross product.

$\langle n_1, n_2, n_3 \rangle$

- $\mathbf{u} \times \mathbf{v}$ is a vector;

- If $\mathbf{u}$ and $\mathbf{v}$ are not linearly dependent, then $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ and its direction is determined by the right hand rule (in particular it is perpendicular to $\mathbf{u}$ and $\mathbf{v}$);



$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

- $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin(\theta)$; $\vec{u} \times \vec{v}$

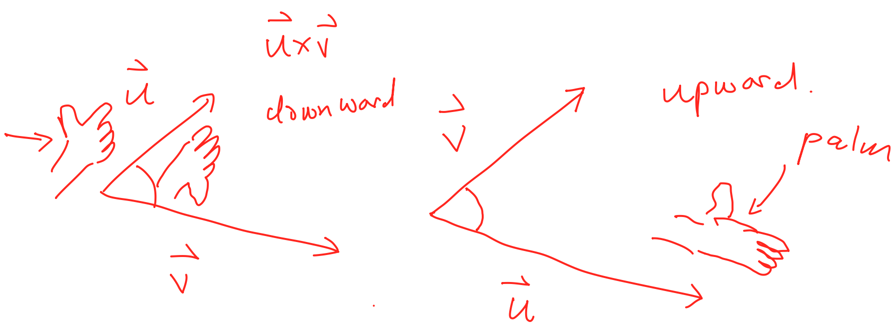
- $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent (so one of them is a scalar multiple of the other).

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$;

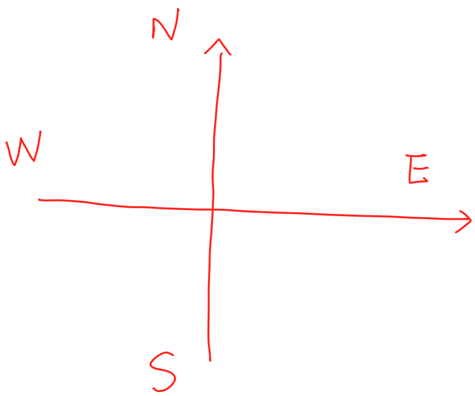
$n_1 x + n_2 y + n_3 z = 0$

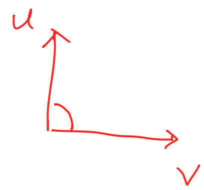
- $\mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v}$;

- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.



Example. If $\mathbf{u}$ points to North, $\mathbf{v}$ points to East, then $\mathbf{u} \times \mathbf{v}$ points to?





downward.

Quiz Question 4. If $\mathbf{u}$ points to West, $\mathbf{v}$ points to North East, then $\mathbf{u} \times \mathbf{v}$ points to?

A. South; B. Upward; C. North; D. Downward