Lecture Notes for 11/7/2023

6.2 General linear transformations

6.3 Isomorphisms

Review of 6.1:

- A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is called a *linear transformation* if it satisfies the following conditions:

  1. For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$;
  2. For any scalar $c \in \mathbb{R}$ and any $\mathbf{x} \in \mathbb{R}^n$, $T(c\mathbf{x}) = cT(\mathbf{x})$.

- (Theorem.) $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a linear transformation if and only if there exists a matrix A of size $k \times n$ such that $T(\mathbf{x}) = A\mathbf{x}$ for any vector $\mathbf{x} \in \mathbb{R}^n$.

- (Theorem.) If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a linear transformation, then its matrix A is given by $A = (T(\mathbf{e}_1), ..., T(\mathbf{e}_n))$. If $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a non-standard basis of $\mathbb{R}^n$ and we know $T(\mathbf{b}_1)$, ..., $T(\mathbf{b}_n)$, then $A = CM_{\mathcal{B}}^{-1}$ where $C = (T(\mathbf{b}_1), ..., T(\mathbf{b}_n))$ and $M_{\mathcal{B}} = (\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n)$.

6.2 General linear transformations

- Let U and V be two general vector spaces. A transformation T from U to V is a linear transformation if it satisfy the same two conditions as in the case of Euclidean vector spaces:

  1. For any $\mathbf{x}$, $\mathbf{y} \in U$, $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$;
  2. For any scalar $c \in \mathbb{R}$ and any $\mathbf{x} \in U$, $T(c\mathbf{x}) = cT(\mathbf{x})$.

Theorem 6.2.1 (Standard matrix of a general linear transformation, strengthened). Let $T : U \longrightarrow V$ be a transformation. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard basis for U and $\mathcal{C} = \{\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_k\}$ be the standard basis for V. Then T is a linear transformation if and only if exists a matrix A of size $k \times n$ such that for any $\mathbf{x} \in U$, $[T(\mathbf{x})]_{\mathcal{C}} = A[\mathbf{x}]_{\mathcal{B}}$. That is, the coordinate vector of $T(\mathbf{x})$ in V with respect to $\mathcal{C}$ equals $A[\mathbf{x}]_{\mathcal{B}}$, where $[\mathbf{x}]_{\mathcal{B}}$ is the coordinate vector of $\mathbf{x}$ in U with respect to $\mathcal{B}$.

$x \in U \Rightarrow [x]_B$ $\qquad T(x) \in V$

$y \in V \Rightarrow [y]_C$ $\qquad [T(x)]_C = A\,[x]_B$

Example 1. If $T : \mathcal{P}_2 \longrightarrow \mathbb{R}_{2\times 2}$ is defined by

$$T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - 3a_2 & 5a_1 \\ 2a_1 + 7a_2 & 0 \end{bmatrix},$$

determine if T is a linear transformation.

$B = \{1, x, x^2\}$

$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} a_0 - 3a_2 \\ 5a_1 \\ 2a_1 + 7a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 5 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Example 2. If $T : \mathbb{R}_{2\times 2} \longrightarrow \mathcal{P}_2$ is defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (1 + 2a - 3d) + (2 - 3b + 4c)x + (a - b + 5d)x^2,$$

determine if T is a linear transformation.

Not linear.

NO

$$\begin{bmatrix} 1 + 2a - 3d \\ 2 - 3b + 4c \\ a - b + 5d \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Quiz Question 1. Determine which of the following transformations is NOT a linear transformation.

A. $T: \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ defined by $T(a_0 + a_1x + a_2x^2) = a_0x^2$;

B. $T: \mathcal{P}_2 \longrightarrow \mathcal{P}_1$ defined by $T(a_0 + a_1x + a_2x^2) = (a_2 - a_1) + (a_0 + 2a_1)x$;

C. $T: \mathcal{P}_1 \longrightarrow \mathcal{P}_3$ defined by $T(a_0 + a_1x) = a_0 + a_1x + a_0x^2 + 3a_1x^3$;

D. $T: \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ defined by $T(a_0 + a_1x + a_2x^2) = (a_0^2 + a_1^2) - 2a_1a_2x + (a_1 - 3a_2)x^2$.

Example 3. Let $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ be the transformation which is the derivative:

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \frac{d}{dx}(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2,$$

is T a linear transformation? If so, what is the matrix of T?

$\uparrow P_2$

$$\begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \longrightarrow a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$

$\mathcal{P}_4 \xrightarrow{T} \mathcal{P}_3 \qquad A = ?$

$$\begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 4a_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Quiz Question 2. If $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_1$ is the derivative:

$$T(a_0 + a_1x + a_2x^2) = \frac{d}{dx}(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$$

find the matrix A for T under the standard basis $\{1, x\}$ for $\mathcal{P}_1$ and $\{1, x, x^2\}$ for $\mathcal{P}_2$.

A. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} a_1 \\ 2a_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Example. Given that $T : \mathbb{R}_{2\times 2} \longrightarrow \mathcal{P}_2$ is a linear transformation and that

$$T\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right) = 3 - 2x + x^2, \quad T\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) = 4x - 2x^2$$

$$T\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right) = 1 + 4x - x^2, \quad T\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = -3 + x.$$

Find the matrix A of T under the standard basis $\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$ for $\mathbb{R}_{2\times 2}$ and the standard basis $\{1, x, x^2\}$ for $\mathcal{P}_2$.

$$A = [T(e_1), T(e_2), T(e_3), T(e_4)]$$

$$= \begin{bmatrix}3 & 0 & 1 & -3\\ -2 & 4 & 4 & 1\\ 1 & -2 & -1 & 0\end{bmatrix}$$

Find $T\left(\begin{bmatrix}2 & -1\\ 5 & 3\end{bmatrix}\right) = \begin{bmatrix}3 & 0 & 1 & -3\\ -2 & 4 & 4 & 1\\ 1 & -2 & -1 & 0\end{bmatrix}\begin{bmatrix}2\\ -1\\ 5\\ 3\end{bmatrix}$

$$= \begin{bmatrix}2\\ 15\\ -1\end{bmatrix} \longrightarrow 2 + 15x - x^2$$

Quiz Question 3. If $T : \mathcal{P}_2 \longrightarrow \mathcal{P}^3$ is a linear transformation and $T(1) = 2x - x^3$, $T(x) = 2 - 3x^2$, $T(x^2) = 1 + x + x^2 + x^3$, find the matrix A of T under the standard basis $\{1, x, x^2\}$ for $\mathcal{P}_2$ and $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$.

A. $\begin{bmatrix} 0 & 2 & 0 & -1 \\ 2 & 0 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & -3 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 2 & -1 \\ 2 & -3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

D. There is nothing here, but you can choose this one to prove that you are wide awake, just not paying attention though.

Now let us discuss Section 6.3.

A transformation $T: U \longrightarrow V$ is 1 to 1 if for any $\mathbf{x} \neq \mathbf{y}$ in U, we have $T(\mathbf{x}) \neq T(\mathbf{y})$. (No two passengers will be going to the same destination.)

Note: An 1-1 transformation is also called an *injective* transformation.

A transformation $T: U \longrightarrow V$ is onto if for any $\mathbf{z} \in V$, there exists at least one $\mathbf{x} \in U$ such that $T(\mathbf{x}) = \mathbf{z}$. (Every destination will have a passenger arriving.)

Note: An onto transformation is also called a *surjective* transformation. A transformation that is both 1-1 and onto is also called a *bijective* transformation or just a bijection.

A linear transformation that is both 1-1 and onto is called an *isomorphism.*

How do we determine whether a linear transformation is 1-1, onto and/or an isomorphism?

This can be determined by examining the matrix A that is associated with T. Let A be of dimension $m \times n$.

- If $\text{rank}(A) = n$ then T is 1-1, otherwise it is not 1-1.
- If $\text{rank}(A) = m$ then T is onto, otherwise it is not onto.
- T is an isomorphism if and only $\text{rank}(A) = m = n$, that is, when A is an invertible matrix.

Example 1. If the matrix A associated with a linear transformation T is of size 4×5, then T cannot possibly be 1-1 since its rank is at most 4, which is less than 5. Similarly, if the matrix A associated with a linear transformation T is of size 7×3, then T cannot possibly be onto since its rank is at most 3, which is less than 7.

Example 2. The linear tranformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an isomorphism.

Example 3. The linear tranformation $T : \mathcal{P}^3 \longrightarrow \mathbb{R}^4$ defined by

$$T\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

is an isomorphism.

$A = I$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = A \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Quiz Question 4. Which of the following transformations is an isomorphism?

A. $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = x^4 + 1$.

B. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$.

C. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

D. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.