Lecture Notes for 11/9/2023

6.3 Isomorphisms, Part 2

6.4 Rank and nullity of a linear transformation

Let us first continue with Section 6.3. Recall:

• $T: U \longrightarrow V$ is 1 to 1 if for any $\mathbf{x} \neq \mathbf{y}$ in U, we have $T(\mathbf{x}) \neq T(\mathbf{y})$.

• $T: U \longrightarrow V$ is onto if for any $\mathbf{z} \in V$, there exists at least one $\mathbf{x} \in U$ such that $T(\mathbf{x}) = \mathbf{z}$.

• A linear transformation that is both 1-1 and onto is called an *isomorphism*.

Let $T: U \longrightarrow V$ be a linear transformation with $A_{m\times n}$ being the matrix associated with it ($n = \dim(U)$ and $m = \dim(V)$) , then

- T is 1-1 if and only if $\text{rank}(A) = n$.
- T is onto if and only if $\text{rank}(A) = m$.
- T is an isomorphism if and only $m = n$ and A is invertible.

$T(x)$ is 1-1 : $T(x) \neq T(y)$ if $x \neq y$

Here are the proofs (for the special case $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$): If $\text{rank}(A) = n$, then the equation system $A\mathbf{x} = \mathbf{0}$ has only the zero solution $\mathbf{x} = \mathbf{0}$ since there are n pivots in the reduced echelon form of A so the equation has no free variables. Thus if $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$, hence $A(\mathbf{x}_1 - \mathbf{x}_2) \neq \mathbf{0}$, that is, $T(\mathbf{x}_1) = A\mathbf{x}_1 \neq A\mathbf{x}_2 = T(\mathbf{x}_2)$. That is, T is 1-1. On the other hand, if T is 1-1, then if $\mathbf{x} \neq \mathbf{0}$, $T(\mathbf{x}) \neq T(\mathbf{0})$. But $T(\mathbf{0}) = A\mathbf{0} = \mathbf{0}$ so $T(\mathbf{x}) = A\mathbf{x} \neq \mathbf{0}$ for any $\mathbf{x} \neq \mathbf{0}$. That means the equation system $A\mathbf{x} = \mathbf{0}$ can only have the zero vector as the solution. Hence the reduced echelon form of A must have n pivots, which means $\text{rank}(A) = n$.

if $y \neq x$, then

$y - x \neq 0$

$T(y-x) \neq 0$

$T(0) = A0 = 0$

$T(y) - T(x) \neq 0$

$T(x) = 0$

$Ax = 0$

If $\text{rank}(A) = m$, then A has m columns that are linearly independent, which form a basis of $\mathbb{R}^m$. It follows that every vector $\mathbf{y} \in \mathbb{R}^m$ is a linear combination of these vectors, hence a linear combination of the column vectors of A. Such a linear combination can be written as $A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. That is, for any $\mathbf{y} \in \mathbb{R}^m$, there exists an $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$. This proves that T is onto. On the other hand, if T is onto, then every vector $\mathbf{y} \in \mathbb{R}^m$ is a linear combination of the column vectors of A. That is, the column space of A is $\mathbb{R}^m$, which has dimension m. Since $\text{rank}(A)$ is the dimension of $\text{Col}(A)$, this proves that $\text{rank}(A) = m$.

Example 1. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear transformation whose associated matrix is A. Given that A has the following echelon form

$$A \longrightarrow \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad T(x) = AX$$

Determine whether A is 1-1, onto, or an isomorphism.

T is not 1-1

T is onto.

not isomorphism.

Example 2. If the linear tranfsormation $T : \boxed{\mathbb{R}^3 \longrightarrow \mathbb{R}^3}$ is onto, is it then also 1-1 hence is an isomorphism? yes.

$$A_{3\times 3} \longrightarrow \begin{bmatrix} 1 & * & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3. What can we say about the linear transformation $T : \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ defined by the derivative? Is it 1-1? onto? isomorphism?

$$A = \left[\; C + 2x + \frac{x^2}{2} + \frac{x^3}{3} \right.$$

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = \boxed{a_1 + 2a_2 x + 3a_3 x^2}$$

$2 + x + x^2$

$5 + x + x^2$

$7 + x + x^2$

$$\begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Quiz Question 1. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a linear transformation whose associated matrix has an echelon form~~s~~ as show below. Identify the correct statement.

$$\underline{A} \longrightarrow \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A. T is 1-1 only;

B. T is onto only;

C. T is an isomorphism;

D. T is neither 1-1 nor onto.

Quiz Question 2. Let A be the matrix associated with a linear transformation T. Given that A has an echelon form as given below

$$\begin{bmatrix} 1 & 2 & 0 & -3 & 1 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which of the following statements is correct about T?

A. T is 1-1 but not onto;

B. T is onto but not 1-1;

C. T is neither 1-1 nor onto;

D. T is both 1-1 and onto (hence it is an isomorphism).

Let us now discuss Section 6.4.

$Ax = 0$

kernal

$T(X) = AX$

Let $T : U \longrightarrow V$ be a linear transformation. The rank of T is the dimension of the range of T, which equals the rank of the matrix A associated with T. Similarly, the nullity of T is the dimension of the null space of T (which is defined as the solution set of the equation $T(\mathbf{x}) = \mathbf{0}$, or $A\mathbf{x} = \mathbf{0}$ with A being the matrix associated with T), and it equals the nullity of the matrix A.

$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$

Note: The range of a linear transformation is also called the *image* of the transformation, and the null space of a linear transformation is also called the *kernel* of the transformation.

Example. Consider the situation in the last quiz question: Let A be the matrix associated with a linear transformation T. Given that A has an echelon form

$$\begin{bmatrix} 1 & 2 & 0 & \cancel{-3} & \cancel{1} \\ 0 & 0 & 1 & \cancel{2} & \cancel{-4} \\ 0 & 0 & 0 & 4 & \cancel{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

we see that $\text{rank}(A) = 4$ and $\text{nullity}(A) = 1$, thus the rank of T is 4 and the nullity of T is 1.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$T : \mathbb{R}^3 \longrightarrow \mathcal{P}_1$ is defined by $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = a_1 + (a_2 - a_3)x$. Find the null space and the range of T (by specifying a basis for each of them), as well as its rank and nullity.

nullity$(T) = 1$

rank$(T) = 2$

$$\begin{bmatrix} a_1 \\ a_2 - a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$x_2 - x_3 = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

T is onto $\Rightarrow$ range$(T) = P_1$

$AX = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Quiz Question 3. The linear transformation $T:\ \mathcal{P}_3 \longrightarrow \mathcal{P}_2$ is defined by the derivative $T\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = a_1 + 2a_2x + 3a_3x^2$. Since the coordinate vector of $a_0 + a_1x + a_2x^2 + a_3x^3$ is $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ under the standard basis $\{1, x, x^2, x^3\}$ of $\mathcal{P}^3$, and the coordinate vector of $a_1 + 2a_2x + 3a_3x^2$ is $\begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}$ under the standard basis $\{1, x, x^2\}$ of $\mathcal{P}^2$, we see that the matrix A associated with T is

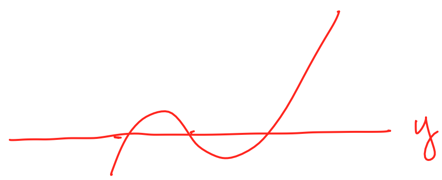
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Find the rank r and nullity n of T.

A. $r = 3,\ n = 4$; B. $r = 1,\ n = 3$; C. $r = 2,\ n = 1$; D. $r = 3,\ n = 1$.

Q & A session on WebWork Assignment #6.

(Potential ones that you may find them difficult: #5, 6, 7, 17, 18, 21, 25)



$$T(2v_1 - 3v_2) = v_1 + v_2$$
$$T(3v_1 - 5v_2) = -2v_1 + 3v_2$$
$$\Rightarrow \quad T(v_1) = ? \qquad T(v_2) = ?$$

$$T(2v_1) + T(-3v_2) = 2T(v_1) - 3T(v_2) = v_1 + v_2$$
$$3T(v_1) - 5T(v_2) = -2v_1 + 3v_2$$

$$2x_1 - 3x_2 = a + b$$
$$3x_1 - 5x_2 = -2a + 3b$$

$$x_1 = \frac{\det(A_1)}{\det(A)}$$

$$A = \begin{bmatrix} 2 & -3 \\ 3 & -5 \end{bmatrix} \quad \det(A) = -10 + 9 = -1$$

$$A_1 = \begin{bmatrix} a+b & -3 \\ -2a+3b & -5 \end{bmatrix} \quad \det(A_1) =$$

#17. $T(M) = \begin{bmatrix} 2 & -7 \\ 0 & 7 \end{bmatrix} M$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2 & -7 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} 2a-7c & 2b-7d \\ 7c & 7d \end{bmatrix}$$

$$\begin{bmatrix} 2a-7c \\ 2b-7d \\ 7c \\ 7d \end{bmatrix} = \overset{A}{\underbrace{\begin{bmatrix} 2 & 0 & -7 & 0 \\ 0 & 2 & 0 & -7 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

#8. $T(\underbrace{a + bx + cx^2}_{f}) = f(8) = \underline{a + 8b + 64c} \quad P_2 \rightarrow \quad +0 \cdot x \quad +0 \cdot x^2$

$$\begin{bmatrix} 1 & 8 & 64 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Quiz Question 4. Let V be a vector space, and $T : V \longrightarrow V$ a linear transformation. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in V such that $T(2\mathbf{u}-5\mathbf{v}) = \mathbf{u}+3\mathbf{v}$ and $T(-3\mathbf{u}+8\mathbf{v}) = -2\mathbf{u}+\mathbf{v}$. Solve for $T(\mathbf{u})$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

A. $-2\mathbf{u}+29\mathbf{v}$; B. $8\mathbf{u}+24\mathbf{v}$; C. $2\mathbf{u}-29\mathbf{v}$; D. $-\mathbf{u}+11\mathbf{v}$.