Before we start, please open your polleverywhere app or a browser and open the page so you will be ready to answer the quiz questions:

`https://pollev.com/yuanandiao611`

1.3 Elementary row operations

How do we attempt to solve a linear equation system such as the following?

$2x_1$ $=7$

$$
\begin{array}{llllllll}
2x_1 & -x_2 & +3x_3 & -4x_4 & +7x_5 & -x_6 & = & 7 \\
3x_1 & +2x_2 & -4x_3 & +x_4 & -3x_5 & +9x_6 & = & -3 \\
-2x_1 & +3x_2 & -x_3 & & +5x_5 & & = & -1 \\
4x_1 & +x_2 & -x_3 & -x_4 & -8x_5 & & = & 12 \\
x_1 & +x_2 & -5x_3 & +9x_4 & -2x_5 & -2x_6 & = & 0 \\
5x_1 & +7x_2 & & +3x_4 & -3x_5 & +8x_6 & = & -5
\end{array}
$$

• Needed: A systematic method that is computation efficient and easy to program.

• How? Keep in mind that whatever we do, we cannot change the equations in such a way that the new equation system will still have the same solutions as the original system of equations.

Quiz Question 1. Choose the operation from the following list that MAY change the solutions to a system of equations.

A. Swap the orders of two equations as they appear in the system;

B. Multiply both sides of an equation with **any** number;

C. Multiply both sides of an equation with any number that is **not zero**;

D. Add the two sides of two equations and use the resulting equation to replace one of these two equations.

$3(2x_1 - 3x_2 + 5x_3 = 4) \longrightarrow \begin{array}{ccc|c} 2 & -3 & 5 & 4 \end{array}$

$6x_1 - 9x_2 + 15x_3 = 12 \qquad \begin{array}{ccc|c} 6 & -9 & 15 & 12 \end{array}$

The operations above that do not change the solutions of the linear equation system can be then defined for the rows of a matrix:

$1 \cdot R_i + R_j \rightarrow R_j$

- Switching two rows ($R_i \longleftrightarrow R_j$)
- Replacing a row by a non-zero multiplying of it ($aR_j \longrightarrow R_j,\ a \neq 0$)
- Replacing a row by adding a (non-zero) multiple of another row ($aR_i + R_j \longrightarrow R_j$)

The three operations defined above are called *elementary row operations*. There are other forms of row operations that are also valid in terms of solving equations, but they are either variations or combinations of the above three elementary row operations and we will stick to these.

Example. Find the matrix resulted from applying the following elementary row operations consecutively in the given order to the matrix

$$\begin{bmatrix}0&0&0&1&5\\3&0&6&9&-3\\0&-2&1&0&0\end{bmatrix} \xrightarrow{R_1\Longleftrightarrow R_2} \begin{bmatrix}3&0&6&9&-3\\0^{0}&0^{0}&0^{0}&1^{-4}&5^{-20}\\0&-2&1&0&0\end{bmatrix}$$

1. $R_1 \longleftrightarrow R_2$;
2. $-4R_2 + R_3 \longrightarrow R_3$;
3. $2R_2 \longrightarrow R_2$.

$$-4\cdot R_2 + R_3 \longrightarrow R_3$$

$$\begin{bmatrix}3&0&6&9&-3\\0&0&0&1&5\\0&-2&1&-4&-20\end{bmatrix}$$

$$\xrightarrow{2R_2\to R_2} \begin{bmatrix}3&0&6&9&-3\\0&0&0&2&10\\0&-2&1&-4&-20\end{bmatrix}$$

$$\begin{array}{c}\begin{matrix}-12&3&-15\end{matrix}\\ \begin{bmatrix}4&-1&5\\1&1&2\\-1&0&7\end{bmatrix}\end{array} \xrightarrow{-3R_1+R_3\to R_3} \begin{bmatrix}4&-1&5\\1&1&2\\-13&3&-8\end{bmatrix}$$

$$\xrightarrow{R_1\leftrightarrow R_2} \begin{array}{c}\begin{matrix}-4&-4&-8\end{matrix}\\ \begin{bmatrix}1&1&2\\4&-1&5\\-13&3&-8\end{bmatrix}\end{array} \xrightarrow{-4R_1+R_2\to R_2} \begin{array}{c}\begin{matrix}13&13&26\end{matrix}\\ \begin{bmatrix}1&1&2\\0&-5&-3\\-13&3&-8\end{bmatrix}\end{array}$$

$$\xrightarrow{13R_1+R_3\to R_3} \begin{bmatrix}1&1&2\\0&-5&-3\\0&16&18\end{bmatrix}$$

Quiz Question 2: Let A be the matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

$3R_3 \longrightarrow R_3$ then

$-R_1 + R_3 \longrightarrow R_3$

If we apply the elementary row operation $3R_3 \longrightarrow R_3$ to A first, followed by the elementary row operation $-R_1 + R_3 \longrightarrow R_3$, identify the resulting matrix from the list below.

A. $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & -3 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ B. $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & -3 \\ 0 & 3 & 0 & 6 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & -3 \\ 1 & 2 & 0 & 6 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 2 & 0 & 6 \\ 1 & 2 & 0 & -3 \\ 0 & 3 & 0 & 6 \end{bmatrix}$

1.4 Echelon forms of a matrix

Key concepts in this section: Echelon form, reduced echelon form, pivots and pivot columns.

$$\begin{bmatrix} 0 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 1 & 0 & 0 \\ 3 & 0 & 6 & 9 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

pivots

$$\begin{bmatrix} 3 & 0 & 6 & 9 & -3 \\ 0 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 5 \\ 0 & -2 & 1 & 0 & 0 \\ 3 & 0 & 6 & 9 & -3 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 6 & 9 & -3 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 6 & 9 & -3 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

-5 -5

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 5 & 0 & 1 & 5 \end{bmatrix}$$

0 5 0 1 5

0 1 1 0 0

0 0 -5 1 5

$$-5R_2 + R_3 \longrightarrow R_3$$

Quiz Question 3. Which of the following matrix is in echelon form?

A. $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 6 & 9 & -3 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 6 & 9 & -3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Reduced echelon form:

- The matrix is in echelon form;
- Each pivot is equal to 1;
- The only non-zero entry in each pivot column is the pivot (which equals 1).

Example:

$$\begin{bmatrix} -1 & 3 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An example of a 4×5 matrix in reduced echelon form:

$$\begin{bmatrix} 1 & ? & ? & ? & ? \\ ? & 1 & ? & ? & ? \\ ? & ? & ? & 1 & ? \\ ? & ? & ? & ? & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & a & 0 & 0 \\ 0 & 1 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quiz Question 4. Which of the following matrix is in reduced echelon form?

A. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 3$

$x_1 = 3$

$x_2 = -1$

$x_3 = 2$

$2x + y = 5$ $\quad y = -2x + 5$

$[2, 1 \quad 5$

$x_1 + \underline{2x_2} = 5$ $\quad [1 \quad 2 \quad 5]$

basic variable

$\underline{x_1 = -2x_2 + 5}$

free variable

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$x_1 + \underline{2x_2} = 2 \qquad x_1 = -2x_2 + 2$$

$$x_3 = -3 \qquad x_3 = -3$$