Lecture Notes for 8/29/2023

There are two free tutoring services available on campus where you can get help for MATH2164:

Link to the PAL program schedule

Link to Mathematics Learning Center

1.5 Solution set of a system of linear equations

The reason that we want to change the augmented matrix of a linear equation system into its reduced echelon form is because from the reduced echelon form we will be able to write down the solutions to the original linear equation system. Furthermore, this allows us to write the solution in a uniform way. Section 1.5 tells us how this is done.

The concept of free variables and basic variables:

If a matrix is an augmented matrix of a linear equation system and is also in reduced echelon form, then the variables corresponding to the pivot columns are called *basic variables* and the variables corresponding to the non-pivot columns are called *free variables*. The solutions of a linear equation system can always be expressed in terms of its free variables (if the system is consistent).

Example 1. $\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & 5 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{array}\right]$

$0 = 3$

The point of this example is: we can tell whether a linear equation system is inconsistent from an echelon form of it.

basic variable. free variable.

Example 2. $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$

$x_1 = 3$

$x_2 = 5$

$x_3 = -4$

$x_4 = 0$

Example 3. $\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$x_1 + 3x_2 + 4x_4 = 4$

$x_1 = -3x_2 - 4x_4 + 4$

$x_1 = -3x_2 - 4x_4 + 4$

$x_2 = x_2$

$x_3 = 2x_4 - 1$

$x_4 = x_4$

$x_5 = 2$

Summarized steps: 1. Verify that the matrix is in reduced echelon form and is consistent; 2. Identify the basic and the free variables; 3. Each free variable equals itself in the solution (that is, it can be any real number); 4. Each basic variable is solved from the row that contains its corresponding pivot by converting that row to an equation and solving for the basic variable which should be the leading variable.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & -1 & 5 \\ 0 & 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 \quad x_3$

$$x_1 = x_1$$
$$x_2 = -2x_4 + x_5 + 5$$
$$x_3 = -x_4 - 2x_5 - 3$$
$$x_4 = x_4$$
$$x_5 = x_5$$

Quiz Question 1. Given that the augmented matrix of a system of linear equation has the reduced echelon form

$$\begin{array}{c} \begin{array}{ccccccc} & 2 & 3 & & 5 & & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 2 \end{array}\right] \end{array}.$$

Identify the free variables. (it may help if you identify the basic variables first.)

A. $x_1,\ x_2,\ x_3$;

B. $x_2,\ x_3,\ x_5$;

C. $x_2,\ x_3,\ x_6$;

D. $x_1,\ x_4,\ x_6$

Quiz Question 2. What is the solution of the system of linear equation whose augmented matrix has the following reduced echelon form?

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 3 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 5 & 0 & 2 \\ 0 & -4 & 1 & 3 & 1 \\ 0 & 0 & 0 & 4 & 3 \end{array}\right]$$

A. $\begin{aligned} x_1 &= 3 - 3x_3 \\ x_2 &= -5 - 2x_3 \\ x_3 &= x_3 \\ x_4 &= 4 \end{aligned}$;

B. $\begin{aligned} x_1 &= 3 \\ x_2 &= -5 \\ x_3 &= 4 \end{aligned}$;

C. $\begin{aligned} x_1 &= 3 - 3x_2 \\ x_2 &= -5 - 2x_2 \\ x_3 &= x_2 \\ x_4 &= 4 \end{aligned}$;

D. $\begin{aligned} x_1 &= 3 - 3x_4 \\ x_2 &= -5 - 2x_4 \\ x_3 &= x_4 \\ x_4 &= 4 \end{aligned}$.

1.6 Gaussian Elimination

• Gaussian Elimination is the procedure that uses the elementary row operations to change a matrix to an echelon form.

$$\begin{bmatrix}2 & -1 & 2 & 10\\ 1 & -2 & 1 & 8\\ 3 & -1 & 2 & 11\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{array}{c} -2 \quad 4 \quad -2 \quad -16 \\ \begin{bmatrix}1 & -2 & 1 & 8\\ 2 & -1 & 2 & 10\\ 3 & -1 & 2 & 11\end{bmatrix}\end{array}$$

$$\xrightarrow{-2R_1+R_2\to R_2} \begin{bmatrix}1^{-3} & -2^{6} & 1^{-3} & 8^{-24}\\ 0 & 3 & 0 & -6\\ 3 & -1 & 2 & 11\end{bmatrix}$$

$$\xrightarrow{-3R_1+R_3\to R_3} \begin{bmatrix}1 & -2 & 1 & 8\\ 0 & 3 & 0 & -6\\ 0 & 5 & -1 & -13\end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2\to R_2} \begin{bmatrix}1 & -2 & 1 & 8\\ 0^{0} & 1^{-5} & 0^{0} & {}^{10}-2\\ 0 & 5 & -1 & -13\end{bmatrix}$$

$$\xrightarrow{-5R_2+R_3\to R_3} \begin{bmatrix}1 & -2 & 1 & 8\\ 0 & 1 & 0 & -2\\ 0 & 0 & -1 & -3\end{bmatrix}$$

$$\xrightarrow{-R_3\to R_3} \begin{bmatrix}1 & -2 & 1 & 8\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 3\end{bmatrix}$$

The key phrase for the order of the "zero out" row operations in the Gaussian elimination process is: *left to right, top to bottom.*

Quiz Question 3. Determine the row operation needed to zero out the entry at row 2 and column 1 of the matrix $\begin{bmatrix} -1 & 3 & 8 \\ 2 & 1 & -2 \end{bmatrix}$.

A. $R_1 \longleftrightarrow R_2$;

B. $-2R_1 + R_2 \longrightarrow R_2$;

C. $\frac{1}{2}R_2 + R_1 \longrightarrow R_1$;

D. $2R_1 + R_2 \longrightarrow R_2$.

$$\begin{bmatrix} -1 & 3 & 8 \\ 0 & 7 & 14 \end{bmatrix} \xrightarrow[\frac{1}{7}R_2 \to R_2]{-1 \times R_1 \to R_1}$$

$$\begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\to R_1]{3 \cdot R_2 + R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$$

$0 \quad 3 \quad 6$

Challenge question: how to zero out the entry at row 3 and column 1 of the matrix $\begin{bmatrix} 4 & 1 & -2 & 0 \\ -5 & 3 & 9 & 1 \\ 3 & 2 & -5 & -2 \end{bmatrix}$ using the pivot 4 without introducing fractions?

$-3 \quad -2 \quad 5 \quad 2$

$$-R_3 + R_1 \longrightarrow R_1$$

$$\begin{bmatrix} 1 & -1 & 3 & 2 \\ -5 & 3 & 9 & 1 \\ 3 & 2 & -5 & -2 \end{bmatrix}$$

1.7 Gauss-Jordan Elimination

Our ultimate goal is to obtain a reduced echelon form, since from which we can easily write out the solutions of the equation system in terms of the free variables. The key point here is that for each pivot column in a matrix already in the echelon form, one MUST use the pivot in this column to zero out the other non-zero entries in the same column. This process is called the Gauss-Jordan Elimination.

$$\begin{bmatrix}2&-1&2&10\\1&-2&1&8\\3&-1&2&11\end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix}1&-2&1&8\\0&1&0&-2\\0&0&1&3\end{bmatrix}$$

$$0 \quad 0 \quad -1 \quad -3$$

$$\xrightarrow{-R_3+R_1\to R_1} \begin{bmatrix}1&-2&0&5\\0&1&0&-2\\0&0&1&3\end{bmatrix} -4$$

$$\xrightarrow{2R_2+R_1\to R_1} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&-2\\0&0&1&3\end{array}\right]$$

Thus if the starting matrix is the augmented matrix of a linear equation system, then its solution is $x_1 = 1$, $x_2 = -2$ and $x_3 = 3$.

The key phrase for the order of the zero out row operations in the Gaussian-Jordan elimination process is: *right to left, bottom to top.*

$$\begin{cases} x_1 + x_2 = 3 \\ x_2 + x_3 = -1 \\ x_3 + x_4 = -3 \\ x_1 + x_4 = 1 \end{cases}$$

$$\begin{array}{ccccc} -1 & -1 & & & -3 \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -3 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_1 + R_4 \to R_4} \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & -1 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_4 \to R_4} \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix} \xrightarrow{-R_3 + R_4 \to R_4}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} \xrightarrow{-R_3 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix}$$

$$\xrightarrow{-R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix}$$

$$x_1 = -x_4 + 1$$
$$x_2 = x_4 + 2$$
$$x_3 = -x_4 - 3$$
$$x_4 = x_4$$

Quiz Question 4. Given the following matrix which is in echelon form already, which of the following operation is NOT a step in the Gauss-Jordan elimination process?

$$
\begin{array}{cccc}
-8 & 4 & -4 & -40 \\
\multicolumn{4}{c}{\left[\begin{array}{cccc} 2 & -1 & 1 & 10 \\ \cancel{0}\,8 & -3 & 4 & -5 \\ 0 & 0 & 2 & 8 \end{array}\right]} \\
 & & -1 & -4
\end{array}
$$

A. $-\frac{1}{2}R_3 + R_1 \longrightarrow R_1$; ✓

B. $-4R_1 + R_2 \longrightarrow R_2$;

C. $\frac{1}{2}R_3 \longrightarrow R_3$;

D. $-2R_3 + R_2 \longrightarrow R_2$.