Lecture Notes for 8/31/2023

2.1 Matrix addition and scalar multiplication

2.2 Matrix multiplication

2.3 Matrix equations and linear systems

Examples.

$$\begin{pmatrix} 7 & -1 & -3 & 9 \\ 2 & -2 & 4 & -1 \\ -5 & 3 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 0 & -2 & 3 & 9 \\ 3 & -1 & 4 & -3 \\ 1 & 7 & 9 & -2 \end{pmatrix} = \begin{pmatrix} 7 & -3 & 0 & 18 \\ 5 & -3 & 8 & -4 \\ -4 & 10 & 9 & 4 \end{pmatrix}$$

$$-2\begin{pmatrix} 7 & -1 & -3 & 9 \\ 2 & -2 & 4 & -1 \\ -5 & 3 & 0 & 6 \end{pmatrix} = \begin{pmatrix} -14 & 2 & 6 & -18 \\ -4 & 4 & -8 & 2 \\ 10 & -6 & 0 & -12 \end{pmatrix}$$

$$x_3\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 - 2x_4 - 6 \\ x_4 + 7 \\ x_3 \\ x_4 \end{pmatrix}$$

Example. As you have seen in some questions of WebWork Assignment 1, the solutions of a linear equation system can be written in a matrix form in terms of matrix addition and scalar multiplication. For example, if the reduced echelon form of a linear equation system is

$$\left[\begin{array}{ccccc|c} 1 & -5 & 0 & 0 & 7 & 2 \\ 0 & 0 & 1 & 0 & -8 & 1 \\ 0 & 0 & 0 & 1 & 1 & -3 \end{array}\right],$$

$x_2 : s_1 \qquad x_5 : s_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5s_1 - 7s_2 + 2 \\ 1\cdot s_1 + 0\cdot s_2 + 0 \\ 0\cdot s_1 + 8s_2 + 1 \\ 0\cdot s_1 - s_2 - 3 \\ 0\cdot s_1 + s_2 + 0 \end{bmatrix} = s_1 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -7 \\ 0 \\ 8 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$$

Quiz Question 1. Which of the following operations is valid?

A. $$-2\begin{pmatrix}0 & -1 & 2 & 9\\ 3 & 7 & -5 & 1\\ 0 & 0 & -4 & -2\end{pmatrix} = \begin{pmatrix}0 & 2 & -4 & -18\\ 3 & 7 & -5 & 1\\ 0 & 0 & -4 & -2\end{pmatrix};$$

B. $$\begin{pmatrix}0 & -1 & 2\\ 3 & 7 & -5\end{pmatrix} - \begin{pmatrix}0 & -1 & 2\end{pmatrix} = \begin{pmatrix}3 & 7 & -5\end{pmatrix};$$

C. $$0\begin{pmatrix}0 & -1 & 2\\ 3 & 7 & -5\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};$$

D. $$\begin{pmatrix}-3 & 2\\ 4 & 9\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}-3 & 2 & 0\\ 4 & 9 & 0\end{pmatrix}.$$

2.2 Matrix multiplication

First, we define the multiplication of a row matrix (from the left side) and a column matrix (from the right side) if the number of rows equals the number of columns:

1×3   3×1

$$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = (1 \times 0 + 2 \times 2 + 0 \times 1) = (4)$$

while

$$\begin{pmatrix} -3 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

would all be undefined.

We can then extend this definition to the multiplication of a row matrix A and a matrix B with several columns where the number of entries in A (namely the number of columns in A) and the number of entries in each column of B (namely the number of rows in B) are equal. For example,

$$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 2 \end{pmatrix}$$

1×3   3×3

$-1+2$

$$\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \qquad 1 \cdot 0 + 2 \cdot 2 + 0 \cdot 1$$

Now we can extend the definition to the multiplication of two matrices A (on the left side) and B (on the right side) when the number of columns in A is equal to the number of rows in B.

AB

BA

$2+1+0 \qquad -4+0+12 \qquad 0+2+3$

$$\begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 2 \\ 5 & 3 & 8 \end{pmatrix}$$

$\underline{2 \times 3} \quad \underline{3 \times 3}$

$\begin{pmatrix} -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$A_{3\times 3} B_{2\times 3}$ $\qquad A_{4\times 5} \cdot B_{5\times 3}$

$A_{2\times 4} \cdot B_{4\times 2} = C_{2\times 2}$

BA

In general, if A has size $n \times m$ and B has size $p \times q$, then AB is only defined if $m = p$ (the number of columns in A equals the number of rows in B). The entry in AB at the i-th row and j-th column is the product of the i-th row in A and the j-th column in B. Furthermore, $A_{n\times m}B_{m\times q}$ is of size $n \times q$. Attention needs to be paid: AB is not the same as BA. When AB is defined, BA may not be defined (and vice versa).

$A_{3\times 3} \quad B_{3\times 3}$ Square

More examples.

$$\begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & -2 \\ 7 & -3 & 17 \end{pmatrix}$$

$-1+8$ $\quad -3 \quad$ $1+16$

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 \end{pmatrix} = \begin{bmatrix} 2 & 6 & -2 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$3 \times 1 \quad 1 \times 3$

Quiz Question 2. Find the entry at row 2 and column 2 of the resulting matrix from the following matrix multiplication.

$3\times 4 \quad 4\times 2$

$$\begin{pmatrix} 2 & 0 & 3 & -2 \\ 0 & 3 & -2 & -1 \\ 5 & 1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \\ 5 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & * \\ & \end{pmatrix}_{3\times 2}$$

A. 10 B. 6 C. 12 D. 14

Quiz Question 3. If the size of A is 3×7 and the size of B is 4×3, which of the following statements is correct?

A. $\cancel{AB}$ is defined and its size is 4×7;

B. $\cancel{BA}$ is undefined;

C. BA is defined and its size is 7×4;

D. BA is defined and its size is 4×7;

$A_{3\times\underline{7}}$ $B_{\underline{4}\times 3}$

$B_{4\times\underline{3}}$ $A_{\underline{3}\times 7}$

But why do we want to define the matrix multiplication like this? Why not the intuitive way like:

$$\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 0 & 12 \end{pmatrix}$$

That is because the matrix multiplication defined in linear algebra here is MUCH more useful than the above "intuitive" definition! It allows us to write a linear equation system in a single equation using matrix multiplication and much more!

$2x_1 + 5x_2 - 3x_3 + x_4 = 7$ can be written as

$$\begin{pmatrix} 2 & 5 & -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \end{pmatrix},$$

$2x = 5$

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 + x_4 &= 7 \\ -x_1 + 2x_2 + 4x_3 - 2x_4 &= 0 \\ 3x_1 \qquad\quad + 2x_3 + 3x_4 &= -2 \end{aligned}$$

B

can be written as

$$\begin{pmatrix} 2 & 5 & -3 & 1 \\ -1 & 2 & 4 & -2 \\ 3 & 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -2 \end{pmatrix},$$ and so on.

$3 \times 4 \quad 4 \times 1$

$$\boxed{Ax = b}$$

In general, any linear equation system with n variables x_1, x_2, ..., x_n and m equations can be written in a simple form (conceptually) $A\mathbf{x} = \mathbf{b}$, where A is a matrix of size $m \times n$, $\mathbf{x}$ is a column matrix with entries x_1, x_2, ..., x_n and $\mathbf{b}$ is a column matrix of size $m \times 1$.

$$\begin{cases} 5x_1 & -4x_2 & +4x_3 & +3x_4 = & 4 \\ -x_1 & +x_2 & +3x_3 & +2x_4 = & 5 \\ 4x_1 & -3x_2 & +7x_3 & +5x_4 = & 9 \\ -3x_1 & +3x_2 & +9x_3 & +6x_4 = & 15 \end{cases}$$

$-1R_2 \to R_2 \qquad -5 \quad 5 \quad 15 \quad 10 \quad -25$

$$\begin{bmatrix} 5 & -4 & 4 & 3 & 4 \\ -1 & 1 & 3 & 2 & 5 \\ 4 & -3 & 7 & 5 & 9 \\ -3 & 3 & 9 & 6 & 15 \end{bmatrix} \xrightarrow[\frac{1}{3}R_4 \to R_4]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & -3 & -2 & 5 \\ 5 & -4 & 4 & 3 & 4 \\ 4 & -3 & 7 & 5 & 9 \\ -1 & 1 & 3 & 2 & 5 \end{bmatrix}$$

$-5R_1 + R_2 \to R_2$

$-4R_1 + R_3 \to R_3$

$R_1 + R_4 \to R_4$

$$\longrightarrow \begin{bmatrix} 1 & -1 & -3 & -2 & 5 \\ 0 & 1 & 19 & 13 & -21 \\ 0 & 1 & 19 & 13 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Quiz Question 4. If we write the following equation system in the matrix form $A\mathbf{x} = \mathbf{b}$, then which of the following is NOT correct?

$$\begin{aligned} x_2 - x_3 &= 2 \\ -3x_1 + 3x_2 + 7x_3 &= -1 \end{aligned}$$

A. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; B. $A = \begin{bmatrix} 0 & 1 & -1 \\ -3 & 3 & 7 \end{bmatrix}$;

C. $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$; D. $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$