Lecture Notes for 9/12/2023

3.1 Introduction to determinants

3.2 Cofactor expansions

Important: The concept of determinants only applies to square matrices. Non-square matrices do not have determinants.

Determinant notations: $\det(A)$, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\underline{ad - bc}$

Example 1. $A = \begin{bmatrix} -4 & 8 \\ 2 & 5 \end{bmatrix}$, then $\det(A) = \begin{vmatrix} -4 & 8 \\ 2 & 5 \end{vmatrix} = (-4) \times 5 - (8) \times (2) = -20 - 16 = -36.$

Example 2. Let $A = \begin{bmatrix} 2 & x \\ -3 & 6 \end{bmatrix}$, find the value of x so that $\det(A) = 0$.

$12 + 3x = 0$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, under the condition that $\det(A) = ad - bc \neq 0$.

What if A is a 3×3 matrix?

Example 3. $A = \begin{bmatrix} 3 & -2 & 0 \\ 5 & 1 & -4 \\ 6 & -1 & 2 \end{bmatrix}$

Example 4. $A = \begin{bmatrix} 2 & 0 & 0 \\ 9 & 3 & 0 \\ -8 & 4 & -5 \end{bmatrix}$

Quiz Question 1. Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 4 \\ -8 & 2 & 3 \end{bmatrix}$

A. 1 B. 3 C. 5 D. 7

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What if A is a 4×4 matrix?

M_{23}

$$A = \begin{bmatrix} -2 & 4 & 0 & 5 \\ 7 & -3 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & -4 & 6 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\det \begin{bmatrix} -2 & 4 & 5 \\ -1 & 1 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

This is when we need to bring Section 3.2 into the discussion about the cofactor expansions.

Definitions of minors and cofactors:

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{vmatrix} -3 & 4 & 0 \\ 1 & 0 & 2 \\ 2 & -4 & 6 \end{vmatrix} = M_{11}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$M_{21} \quad M_{22} \quad M_{23} \quad M_{24}$

$C_{21} = -M_{21}$

$C_{22} = M_{22}$

$C_{23} = -M_{23}$

$C_{24} = M_{24}$

$$M_{41} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

We now define the cofactors C_{ij} to be M_{ij} if $i+j$ is even, and C_{ij} to be $-M_{ij}$ if $i+j$ is odd. Examples: $C_{11} = M_{11}$, $C_{23} = -M_{23}$, $C_{41} = -M_{41}$. This can be expressed using a single formula $C_{ij} = (-1)^{i+j} M_{ij}$.

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$$

C_{23}

Quiz Question 2. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 4 \\ -8 & 2 & 3 \end{bmatrix}$, what is C_{23}?

A. 2 B. -2 C. 3 D. -3

$$M_{23} = \begin{vmatrix} 1 & 0 \\ -8 & 2 \end{vmatrix} = 2 - 0 \cdot (-8) = 2$$

Steps to compute $\det(A)$ if A has size 4×4:

Step 1. Choose any row or any column. Say the third row in the example below.

Step 2. For each entry in that row, multiply it with its corresponding cofactor.

Step 3. Summing over the results in Step 2.

Example: $A = \begin{bmatrix} -2 & 4 & 0 & 5 \\ 7 & -3 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & -4 & 6 \end{bmatrix}$

Say we choose the third row for the cofactor expansion. We have

$a_{31} = -1$, $a_{32} = 1$, $a_{33} = 0$, $a_{34} = 2$ and

$C_{31} = M_{31} = \begin{vmatrix} -4 & 0 & 5 \\ -3 & 4 & 0 \\ 2 & -4 & 6 \end{vmatrix} = -76,$ $-1 = a_{31}$

$C_{32} = -M_{32} = -\begin{vmatrix} -2 & 0 & 5 \\ 7 & 4 & 0 \\ 0 & -4 & 6 \end{vmatrix} = 188,$ $1 = a_{32}$

$C_{33} = M_{33} = \begin{vmatrix} -2 & 4 & 5 \\ 7 & -3 & 0 \\ 0 & 2 & 6 \end{vmatrix} = 62,$ $0 = a_{33}$

$C_{34} = -M_{34} = -\begin{vmatrix} -2 & 4 & 0 \\ 7 & -3 & 4 \\ 0 & 2 & -4 \end{vmatrix} = -8,$ $a_{34} = 2$

so $\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34} = (-1) \cdot (-76) + 1 \cdot 188 + 0 \cdot 62 + 2 \cdot (-8) = 248$

$\vec{u}$

$\vec{v}$

vector : one column matrix

Quiz Question 3. Given that A has size 4×4, $a_{11} = 3$, $a_{21} = 2$, $a_{31} = -5$, $a_{41} = 2$ and $C_{11} = 4$, $C_{21} = -2$, $C_{31} = 0$, $C_{41} = -1$, find $\det(A)$.

A. 12 B. 6 C. 1 D. 18

$3 \cdot 4 + 2 \cdot (-2) + (-5) \cdot 0 + 2 \cdot (-1)$

$= 12 - 4 - 2$

The above procedure allows us to compute $\det(A)$ if A is 4×4. The cofactor expansion procedure can now be performed on 5×5 matrices since we now know how to compute the determinants of 4×4, and so on. At least in theory, we know how to compute the determinant of a square matrix of any size. In some special cases, the computation following this procedure is simple.

For example, if A has size 5×5, and we have found that $a_{31} = -2$, $a_{32} = 0$, $a_{33} = 3$, $a_{34} = 1$, $a_{35} = 0$. Now we would be able to compute C_{31}, C_{32}, C_{33}, C_{34}, and C_{35} since they involve the computation of the determinants of 4×4 matrices. Say after calculation we found that $C_{31} = 1$, $C_{32} = 2$, $C_{33} = 0$, $C_{34} = 0$, and $C_{35} = -4$

then

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34} + a_{35}C_{35}$$

Special case: if A is a triangular matrix.

$$A = \begin{bmatrix} -3 & 4 & 0 & 5 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

0

$-3 \cdot \boxed{C_{11}}$

$-3 \cdot 3 \cdot 2 \cdot (-2)$

Quiz Question 4. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 14 & 20 & 35 & 45 \\ 0 & 3 & -3 & 0 & 7 \\ 0 & 0 & -2 & 12 & 8 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

A. 0 B. 1 C. -12 D. 45

$\underline{1 \cdot C_{11}} + \underline{0 \cdot C_{21} + 0 \cdot C_{31}}$

$$\begin{vmatrix} -10 & 0 & -1 \\ -9 & -4 & 3 \\ 12 & -5 & 8 \end{vmatrix}$$