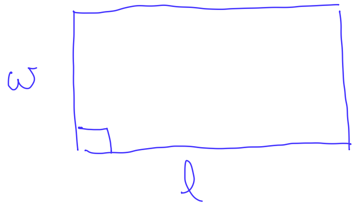
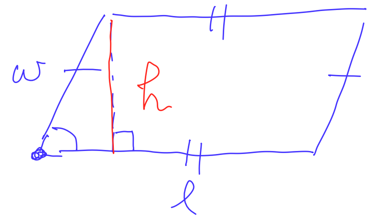
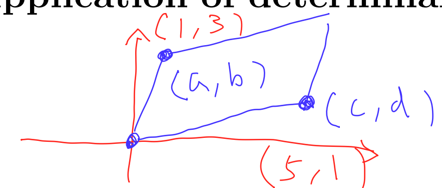


$A = w \cdot l$



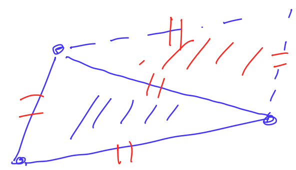
Lecture Notes for 9/14/2023

3.3 Application of determinants: Area and volume



Consider a parallelogram in the xy-coordinate plane with one vertex at the origin, and two adjacent vertices at (a, b) and (c, d). Then the area of the parallelogram is given by

$$|\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}|$$



Since the area of the triangle with vertices $(0, 0)$, (a, b) and (c, d) is half of the parallelogram, the area of the triangle is given by the formula

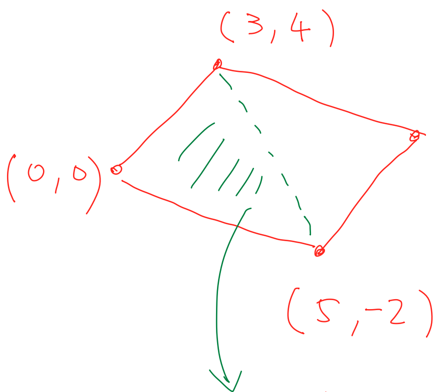
$$\frac{1}{2}|\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}|$$

Note: points in the plane can be expressed as the column matrices. For example, $(0, 0)$, (a, b) and (c, d) can be written as $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$, and $\begin{bmatrix} c \\ d \end{bmatrix}$. This does not change the formula for the area since

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc.$$

$\det(A) = \det(A^T)$

Examples.



$A = ?$

$$A = \left| \det \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \right|$$

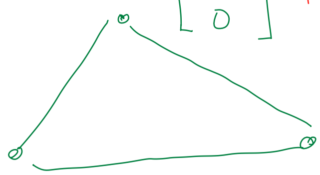
$$= |-6-20| = 26$$

area of triangle $= \frac{1}{2} \cdot 26 = 13$

Quiz Question 1. Find the area of the triangle with vertices $(0,0)$, $(3,-2)$ and $(-5,8)$.

A. 14 B. 7 C. 34 D. 17

What if none of the vertices of the triangle is the origin?



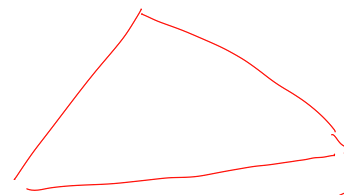
$$\begin{bmatrix}3\\0\end{bmatrix}+\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\begin{bmatrix}-4\\5\end{bmatrix}+\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}-1\\1\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix}$$

$\longrightarrow$

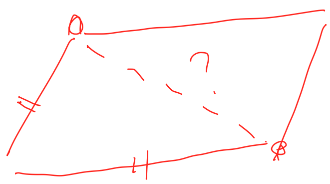
$$\begin{bmatrix}2\\1\end{bmatrix}$$



$$\begin{bmatrix}0\\0\end{bmatrix} \qquad \begin{bmatrix}-5\\6\end{bmatrix}$$

$$A = \frac{1}{2}\left|\det\begin{bmatrix}2 & -5\\1 & 6\end{bmatrix}\right|$$

$$= \frac{1}{2}\left|12+5\right| = \frac{17}{2}$$

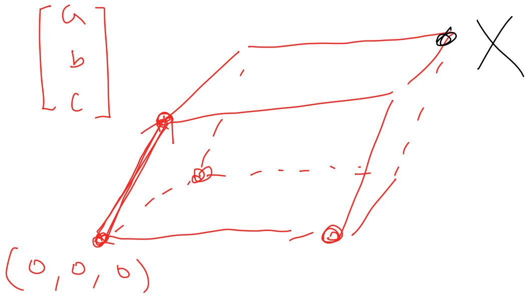


$(x_1, y_1) \quad (x_2, y_2)$

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Quiz Question 2. Find the area of the triangle with vertices $(1, -3)$, $(4, 4)$ and $(0, -6)$.

A. 2 B. 8 C. 1 D. 4



The volume of a rectangular box with side lengths of u, v and w is $V = uvw$. In the plane, a rectangle generalizes to a parallelogram. In the 3-d space, a rectangular box generalizes to a solid shape called *parallelepiped*. If the parallelepiped is placed in the xyz-coordinate system with one vertex at $(0,0,0)$, and the other three adjacent vertices at (a,b,c), (d,e,f) and (g,h,i), then the volume of the parallelepiped is given by $|\det(A)|$, where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Similar to the case of a parallelogram, we can express the vertices using column matrices as well:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}, \begin{bmatrix} g \\ h \\ i \end{bmatrix}$$

This is because we also have

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}.$$

Example 1. Find the volume of the parallelepiped defined by the column vectors

$$\begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}.$$

$$\left| \begin{matrix} 2 & 1 & -3 & 2 & 1 \\ 3 & 0 & 2 & 3 & 0 \\ -5 & -1 & 4 & -5 & -1 \end{matrix} \right.$$

$$0 - 10 + 9 - 0 - (-4) - 12$$

$$= -1 + 4 - 12 = -9$$

$$V = |-9| = 9$$

Example 2. Determine the values of x so that the volume of the parallelepiped defined by the following column vectors is 8:

$$\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$x + 7 = 8$$

$$x + 7 = -8$$

$$\left| \begin{matrix} x & 2 & -1 & x & 2 \\ \boxed{0} & 1 & 3 & 0 & 1 \\ \boxed{1} & 0 & 1 & 1 & 0 \end{matrix} \right.$$

$$|x + 7| = 8$$

$$x = 1, \text{ or } x = -15$$

$$x \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = x + 7$$

Quiz Question 3. Find the values of x so that the volume of the parallelepiped defined by the following column vectors is 6:

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \qquad \begin{bmatrix}-3\\2\\-1\end{bmatrix}, \qquad \begin{bmatrix}0\\x\\2\end{bmatrix}$$

$$\begin{matrix}1 & -3 & 0 & 1 & -3\\ 0 & 2 & x & 0 & 2\\ 1 & -1 & 2 & 1 & -1\end{matrix}$$

$4 - 3x + 0 - (-x) + 0$

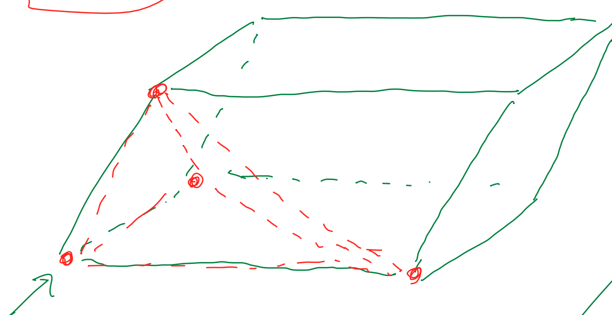
$4 - 3x + x$

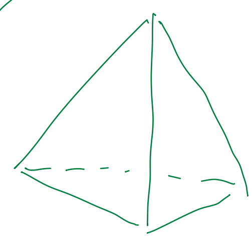
$= |4 - 2x| = 6$

$4 - 2x = 6$

$4 - 2x = -6$

A. $x = -1$; B. $x = -1,\ 5$; C. $x = 3,\ 5$; D. No solution.





The volume of a tetrahedron.

$\frac{1}{6} \cdot$ Vol of parellelepiped.

Quiz Question 4. Find the volume of the tetrahedron defined by the column vectors

$$\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}.$$

A. 1; B. 4; C. 12; D. 24.

$20 + 0 + 8$

$$\begin{matrix} 1 & 0 & -1 & 1 & 0 \\ -4 & 4 & 2 & -4 & 4 \\ 0 & 2 & 5 & 0 & 2 \end{matrix} \qquad - \ ?$$

Practice for Test 1 is open. In preparation for the review session on Tuesday next week, please let me know which questions or what types of questions that you would like me to go over during the review session. I will put more emphasis on the ones that receive more requests.

$$\begin{array}{cccc} & & & \\ \left[\begin{matrix} 1 & 3 & -5 & -1 \end{matrix}\right] \\ \uparrow & x_2 & x_3 & x_4 \end{array} \qquad x_1 = -3x_2 + 5x_3 + x_4$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ \\ \\ \end{bmatrix}$$

$$x_2 = \underline{1} \cdot x_2 + \underline{0} \cdot x_3 + \underline{0} \cdot x_4$$