Lecture Notes for 9/28/2023

3.5 Invertibility and determinants

3.6 Cramer's rule

Theorem 3.5.1. Let A be a square matrix. Then A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = 1/\det(A)$.

Theorem 3.5.4. Let A be an $n \times n$ matrix, then the follow statements are equivalent:

1. A is invertible;
2. The reduced row echelon form of A is the identity matrix I_n;
3. A has n pivot columns (or n pivots, or n pivot positions);
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$;
5. There exists an $n \times n$ matrix C such that $CA = I_n$;
6. There exists an $n \times n$ matrix D such that $AD = I_n$;
7. The transpose A^T of A is invertible;
8. $\det(A) \neq 0$.

We will be using this result later (many times in fact) so try to read it a few times to keep it in your memory for a while, it will be handy!

$$\det(A^{-1}B) = \left(\frac{1}{\det(A)}\right)\det(B)$$

$$\times \det(A+B) = \det(A) + \det(B) \times ???$$

One more important property of determinants (this is covered in WebWork and will be covered in the next test):

Let A and B be two $n \times n$ matrices that are identical except for one row (or one column). Say the first row of A and B may be different, that is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$(a_{11}+b_{11})C_{11} + \cdots$

If C is the matrix obtained from A by adding the first row of B to the first row of A:

$a_{11}C_{11}$

$b_{11}C_{11}$

$$C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

then $\det(C) = \det(A) + \det(B)$.

Example. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$, $\begin{vmatrix} a & b & c \\ 2 & 3 & -1 \\ g & h & i \end{vmatrix} = -7$, find $\begin{vmatrix} a & b & c \\ 3d-4 & 3e-6 & 3f+2 \\ g & h & i \end{vmatrix}$.

$$= \begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ -4 & -6 & 2 \\ g & h & i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - 2\begin{vmatrix} a & b & c \\ 2 & 3 & -1 \\ g & h & i \end{vmatrix}$$

$$= 3 \cdot 4 - 2(-7) = 12 + 14 = 26$$

Example. If $\begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 3 & 3 \end{vmatrix} = 12$, $\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{vmatrix} = -5$, find $\begin{vmatrix} a & b & c \\ d & e & f \\ -3 & -4 & -5 \end{vmatrix}$.

$$3\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{vmatrix} = 12 \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{vmatrix} = 4$$

$$-\begin{vmatrix} a & b & c \\ d & e & f \\ 3 & 4 & 5 \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ 2+1 & 2+2 & 2+3 \end{vmatrix} = -\left(2\overset{4}{\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{vmatrix}} + \underset{-5}{\begin{vmatrix} - & - & - \\ - & - & - \\ 1 & 2 & 3 \end{vmatrix}}\right)$$

$$= -(2 \cdot 4 + (-5)) = -3$$

$$\overset{-2}{x}(1\ 1\ 1) + \overset{-1}{y}(1\ 2\ 3) = (-3, -4, -5)$$

$x + y = -3$ $\qquad$ $x+y=-3$

$-(x+2y=-4)$ $\qquad$ $-x-2y=4$ $\qquad$ $-y=1$

$y=-1$

$-2\begin{vmatrix} a & b & c \\ d & e & f \\ -2 & 3 & 1 \end{vmatrix} +$

Quiz Question 1. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 1 & 4 \end{vmatrix} = -3 \text{ and } \begin{bmatrix} a & b & c \\ d & e & f \\ -2 & 3 & 1 \end{bmatrix} = 4, \text{ find } \begin{vmatrix} a & b & c \\ d & e & f \\ 4 & -5 & 2 \end{vmatrix}. = \begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 1 & 4 \end{vmatrix}$$

A. -11; B. 5; C. 1; D. -2.

$x - 6 = -5$

11

$$x(0\ 1\ 4) + \overset{-2}{y}(-2\ 3\ 1) = (4\ -5\ 2)$$

$0 + -2y = 4$

$y = -2$

Cofactor matrix and adjoint matrix of A:

Let A be an $n \times n$ matrix, with C_{ij} being the cofactor of the entry a_{ij} in A. Then

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the cofactor matrix of A and the transpose of it is called the adjoint matrix of A, denoted by adj(A):

$$C^T = \text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$A \cdot C^T = C^T A = \det(A)\, I$

$\left(\boxed{\frac{1}{\det(A)} C^T}\right) A = I$

Theorem 3.5.2. Let A be a square matrix and $adj(A)$ the adjoint matrix of A, then $A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) I_n$. Thus if $\det(A) \neq 0$, then $\boxed{A^{-1} = \frac{1}{\det(A)} \text{adj}(A)}$.

The proof of this theorem is given in the book.

$C^T = \det(A)\, A^{-1}$

If A is a 4×4 matrix such that

$$A^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$\det(A^{-1}) = 2$

$\det(A) = \frac{1}{2}$

(so $\det(A) = 1/\det(A^{-1}) = 1/2$), find the cofactor matrix of A and use it to find the cofactors C_{23} and C_{44}.

$C^T = \frac{1}{2} \begin{bmatrix} \; \end{bmatrix}$

$$C = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 5 & 3 & -2 \\ 1 & 4 & -2 & 1 \end{bmatrix}$$

$C_{23} = 0 \qquad C_{44} = \frac{1}{2}$

$C_{34} = \frac{1}{2}(-2) = -1$

Quiz Question 2. Continue from the last example, where 4×4 matrix such that

$$A^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

(so $\det(A) = 1/\det(A^{-1}) = 1/2$), find the cofactor C_{14}.

A. $-1/2$; B. 0; C. $1/2$; D. 1.

$$A^{-1} = \frac{1}{\det(A)} \cdot C^T \qquad C^T = \det(A)\, A^{-1}$$

$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ & & & \\ & & & \end{bmatrix} \qquad C = \det(A)\,(A^{-1})^T$$

3.6 Cramer's rule

$A^{-1}Ax = A^{-1}b \quad x = A^{-1}b$

$x_j = \frac{\det(A_j)}{\det(A)}$

Theorem 3.6.1. Consider the linear equation system $A\mathbf{x} = \mathbf{b}$ with n equations and n variables, so that the coefficient matrix A is of size $n \times n$. Let A_j be the matrix obtained from A by replacing its j-th column with $\mathbf{b}$, then if $\det(A) \neq 0$, the equation has a unique solution which is given by $x_j = \frac{\det(A_j)}{\det(A)}$ for $1 \leq j \leq n$.

$x_1 \quad x_2 \quad \ldots \quad x_n$

Note that under this setting A is an $n \times n$ matrix, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots$

Example. Find the solution of the equation system

$$2x_1 + 5x_2 = a, \quad -x_1 + 3x_2 = b$$

where a and b are some constant numbers. Express your answer in terms of a and b.

$$\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$\vec{x} \quad \vec{b}$

$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$ $\quad \det(A) = 6+5 = 11$

$\det(A_1) = \begin{vmatrix} a & 5 \\ b & 3 \end{vmatrix} = 3a - 5b$

$\det(A_2) = \begin{vmatrix} 2 & a \\ -1 & b \end{vmatrix} = 2b - (-a) = a + 2b$

$x_1 = \frac{1}{11}(3a - 5b), \quad x_2 = \frac{1}{11}(a + 2b)$

Quiz Question 3. Consider the linear equation system $A\mathbf{x} = \mathbf{b}$ with n equations and n unknowns. If $\det(A) = 0$, then which of the following statement is always true?

a. We can still apply Cramer's rule to solve the equation;

b. The equation is definitely inconsistent;

c. The equation is consistent and has a unique solution;

d. The equation is either inconsistent, or is consistent with infinitely many solutions.

Quiz Question 4. Consider the linear equation system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 4 & 3 & 1 \\ 1 & -5 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -17 \end{bmatrix}.$$

Given that $\det(A) = -19$, find x_2.

x_1

a. The solution does not exist; b. 2; c. -2; d. $2/19$.

$$\begin{vmatrix} -2 & 0 & 0 \\ 4 & 2 & 1 \\ 1 & -17 & 1 \end{vmatrix} \qquad x_2 = \frac{\det(A_2)}{\det(A)}$$

$$= -38/-19 = 2$$