Lecture Notes for 9/5/2023

Review: In 2.2 we defined matrix multiplication. A matrix $A_{m\times n}$ can be multiplied to a matrix B is B has size $n \times q$, and AB has size $m \times q$. Why do we define the multiplication this way? Here is a reason.

Consider the following linear equation system:

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 + x_4 &= b_1 \\ -x_1 + 2x_2 + 4x_3 - 2x_4 &= b_2 \\ 3x_1 \qquad + 2x_3 + 3x_4 &= b_3 \end{aligned}$$

This equation can be written in the matrix form as

$$\begin{bmatrix} 2 & 5 & -3 & 1 \\ -1 & 2 & 4 & -2 \\ 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The problem here is that we do not know what are the values of b_1, b_2 and b_3. However in this example, b_1, b_2 and b_3 are the solution of the following equation system:

$$\begin{aligned} b_1 - 3b_2 + 4b_3 &= 3 \\ 2b_1 + b_2 - b_3 &= -7 \\ -3b_1 + 2b_2 + b_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & 1 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -3 & 1 \\ -1 & 2 & 4 & -2 \\ 3 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}$$

which can be written in the matrix form

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & 1 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}.$$

How would you solve for x_1, x_2, x_3, x_4?

We had observed that in general $AB \neq BA$: sometimes BA is not defined even when AB is defined, sometimes AB and BA are both defined but have different sizes, and even in the case that A and B are both $n \times n$ square matrices so AB and BA are also $n \times n$ matrices, AB and BA are still not equal in general. But other than this, most rules we know about number multiplication are actually still true for matrix multiplication:

- $A(BC) = (AB)C$ (Associative property of matrix multiplication)
- $A(B + C) = AB + AC$ (Left distributive property)
- $(A + B)C = AC + BC$ (Right distributive property)
- $r(AB) = (rA)B = A(rB)$ (Associative property of scalars in matrix multiplication)
- $A_{m\times n}I_n = I_m A_{m\times n} = A$ (Identity for matrix multiplication)

In the above I is the identity matrix, for example $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and so on. Here is an example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \\ 7 & -4 & 3 \\ 0 & 5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ 7 & -4 & 3 \\ 0 & 5 & -8 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 1 \\ 7 & -4 & 3 \\ 0 & 5 & -8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ 7 & -4 & 3 \\ 0 & 5 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 4 \end{bmatrix}$$

$2\times \underline{2\ 2}\times 3$

$(AB)^T$ $A^T B^T$

One other important formula:

- $(AB)^T = B^T A^T$ (transpose of multiplication)

$A_{3\times 4} \quad B_{4\times 5}$

Also, some matrices are more special than the others, and they deserve to have their own names for easier reference in the future. Here are a few.

$A^T_{4\times 3} \quad B^T_{5\times 4}$

Square matrices: $A_{n\times n}$ (matrices with the same number of rows and columns)

$B^T_{5\times 4} \quad A^T_{4\times 3}$

5×3

Triangular matrices:

$$\begin{bmatrix} -2 & 3 & 1 \\ 0 & -4 & 3 \\ 0 & 0 & -8 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{ (upper triangular)}$$

$$\begin{bmatrix} -2 & 0 \\ 2 & -8 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 \end{bmatrix} \text{ (lower triangular)}$$

Diagonal matrices:

$$\begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -8 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Quiz Question 1. Let A, B and C be square matrices of the same size $n \times n$. Which of the following statement regarding matrix multiplication is true?

$A(B+C)$

A. $A(B-C) = AB - AC$; $\rightarrow A(B+(-1)C) = AB + A(-1)C$

B. $(A+B)^2 = A^2 + 2AB + B^2$;

$(A+B)(A+B)$

C. $(A+B)(A-B) = A^2 - B^2$; X

$A^2 + AB + BA + B^2$

D. $(AB)^2 = A^2B^2$.

$(AB)(AB) = A(BA)B$

The point of this quiz question: Sometimes a familiar formula such as $(a+b)(a-b) = a^2 - b^2$ with real numbers uses the fact $ab = ba$ implicitly, which means we do not have $(A+B)(A-B) = A^2 - B^2$ for matrices.

2.4 Inverse of a matrix

Let A be a square matrix. A matrix B is called the *inverse* of A if B has the same size as A and $AB = BA = I$.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$0 + 0 - 3$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & & \\ & & \\ & & \end{bmatrix}$$

NOT I

$-2 + 2 + 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of invertible matrices:

$(AB)^{-1}$

(1) If A is invertible, then A^{-1} is unique;

I

(2) If A and B are both $n \times n$ invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$; $A^{-1}B^{-1}$

$\underline{AB\,B^{-1}\cdot A^{-1}}$

(3) If A is invertible and $k \neq 0$ is a scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

$ABA^{-1}B^{-1}$

(4) If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$.

Proof. (1) Assume that B and C are both inverses of A. Then by definition $AB = BA = I$ and $AC = CA = I$. Multiply both sides of the equation $AB = I$ by C (from left): $C(AB) = CI$. The right side is just $CI = C$. The left side is (by associativity of matrix multiplication) $C(AB) = (CA)B = IB = B$. Thus $B = C$. So we have shown any two inverse matrices of A must in fact be the same. So the inverse of A is unique.

$(kA)(\frac{1}{k}A^{-1})$

$=$

(2) By direct verification:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Similarly $(B^{-1}A^{-1})(AB) = I$. So $B^{-1}A^{-1} = (AB)^{-1}$.

(3) By the distribution of scalar multiplication and matrix multiplication,

$$(kA)(\frac{1}{k}A^{-1}) = (k\frac{1}{k})(AA^{-1}) = 1 \cdot I = I.$$

Similarly $(\frac{1}{k}A^{-1})(kA) = I$. That is, $(kA)^{-1} = \frac{1}{k}A^{-1}$.

(4) Taking the transpose on all sides of $A^{-1}A = AA^{-1} = I$, we get $A^T(A^{-1})^T = (A^{-1})^TA^T = I^T = I$. So $(A^T)^{-1} = (A^{-1})^T$.

Question: how do we know when a matrix is invertible and how to find the inverse?

If A is a 1×1 matrix then $A = [a]$, $A^{-1} = [a^{-1}]$ if $a \neq 0$, otherwise A^{-1} does not exist.

If A is a 2×2 matrix, that is, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad - bc \neq 0$. If $ad - bc = 0$, A^{-1} does not exist.

$ad - bc$

For example, if

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix},$$

$ad - bc = 3 \cdot 2 - 4(-2)$

$= 6 + 8 = \underline{14} \neq 0$

then

$$A^{-1} = \frac{1}{14} \begin{bmatrix} 2 & -4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Quiz Question 2. Which of the following matrices is NOT invertible?

A. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ B. $\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 2 & -7 \\ -4 & 14 \end{bmatrix}$ D. $\begin{bmatrix} 0 & 3 \\ -1 & 3 \end{bmatrix}$

For a general $n \times n$ matrix A with $n \geq 3$, how do we find A^{-1}?

Answer: by using Gauss-Jordan elimination on the augmented matrix $[\ A \mid I\ ]$ to reduce it to a reduced echelon form. If the reduced echelon form is I, then the right side is A^{-1}. Otherwise A^{-1} does not exist.

Example 1. $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 2 \end{bmatrix}$

$$\begin{array}{cccccc} -5 & & & -5 & & \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 5 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-5R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 3 & 2 & -5 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccccc} & -3 & & -6 & -3 & \\ \end{array}$$

$$\xrightarrow{-3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -11 & -3 & 1 \end{array}\right] \quad \frac{1}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{11}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \quad A^{-1}$$

Example 2. $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & -6 \\ 0 & 1 & 1 \end{bmatrix}$

$$\begin{array}{c} \begin{matrix} -2 & -6 & 0 & -2 & 0 & 0 \end{matrix} \\ \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -6 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -6 & -6 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -6 & -6 & -2 & 1 & 0 \end{bmatrix} \longrightarrow$$

$0 \quad 0 \quad 0$ No inverse.

Quiz Question 3. Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

Given a linear equation system $A\mathbf{x} = \mathbf{b}$ with A being a square matrix, then in the case that *we know* the inverse of A, the equation system is easy to solve: we multiply A^{-1} (from left) to both sides of the equation: $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$. The left side is always $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I \cdot \mathbf{x} = \mathbf{x}$, so the solution of the equation system is simply $\mathbf{x} = A^{-1}\mathbf{b}$.

$A_{n \times n} \, X_{n \times 1}$

For example, if the equation is

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

$A x = b_{n \times 1} \quad A_{n \times n}$

$A^{-1}(Ax) = A^{-1} b$

and we know that $A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, then

$x = A^{-1} b$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Notice that we don't even have to know A in solving for $\mathbf{x}$. Of course we can find A from A^{-1} if we want to, but why bother if we only want to find $\mathbf{x}$?

Quiz Question 4. Assume that A is a 3×3 matrix and we know its inverse matrix is

$$A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

$x = A^{-1} b$

Solve the equation system

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}. \quad = \begin{bmatrix} 3 \\ -6 \\ \end{bmatrix}$$

A. $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$; B. $\begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}$; C. $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$; D. $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.