Lecture Notes for 9/7/2023

2.5 Solving a system using an inverse matrix

Given a linear equation system $A\mathbf{x} = \mathbf{b}$ with A being a square matrix, then in the case that *we know* the inverse of A, the equation system is easy to solve: we multiply A^{-1} (from left) to both sides of the equation: $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$. The left side is always $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I \cdot \mathbf{x} = \mathbf{x}$, so the solution of the equation system is simply $\mathbf{x} = A^{-1}\mathbf{b}$.

$$Ax = b_1$$
$$\vdots$$
$$Ax = b_{10}$$

For example, if the equation is

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix},$$

and we know that $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & 3 \\ 2 & 1 & 0 \end{bmatrix}$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix}.$$

Notice that we don't even have to know A in solving for $\mathbf{x}$. Of course we can find A from A^{-1} if we want to.

2.6 Elementary matrices

Review: the procedure to find the inverse matrix of a given matrix. For example, how do we find the inverse of the matrix $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{bmatrix}$?

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{4R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 1 \end{bmatrix}$$

At the end of this we conclude that $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}$.

But HOW DO WE KNOW THIS??? The elementary matrices provide us the main tool in explaining this.

An *elementary matrix* is a matrix obtained by applying one of the three elementary row operations to the identity matrix.

Examples. Here are some elementary matrices.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{4R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = E_2$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = E_4$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix} = E_5$$

Quiz Question 1. Which of the following is the 4×4 elementary matrix obtained by the row operation $5R_4 + R_2 \longrightarrow R_2$?

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Given a square matrix, how do we know that it is an elementary matrix or not?

Answer: it is an elementary matrix if it can be changed back to the identity matrix by ONE elementary row operation! And you should be able to identify which elementary row operation was used to obtain the elementary matrix. Look at the examples we just did:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{4R_1+R_3 \\ \to R_3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{-4R_1+R_3 \\ \to R_3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The following matrices are NOT elementary matrices since they CANNOT be obtained from an identity matrix by a SINGLE elementary row operation.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Quiz Question 2. Which of the following is NOT an elementary matrix?

A. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$R_2 + (-1)R_3 \to R_3$

Why bother with elementary matrices?

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \overset{E_1A}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}$$

$$\xrightarrow{4R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 1 \end{bmatrix}$$

$$E_1A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E_1 \cdot I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \overset{E_1A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -4 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2(E_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$\boxed{(E_3E_2E_1)}A = E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_3E_2E_1 \quad I$

$$E_3(E_2E_1) = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$(A \quad I) \quad I \qquad \underline{E_k \cdots E_1} = A^{-1}$

$(E_1 A \quad E_1 I) \longrightarrow (E_2 E_1 A,\ E_2 E_1 I)$

$\cdots \quad (\underbrace{E_k \cdots E_1 A},\ \underline{E_k \cdots E_1 I})$

In the above example, performing the 3 elementary row operations on A is the same as multiplying the elementary matrices E_1, E_2 and E_3 consecutively to A: $E_3E_2E_1A$, since we are also applying the same operations to I (on the right side of A in the augmented matrix), the right side of this augmented matrix is precisely $E_3E_2E_1I = E_3E_2E_1$. We saw that $E_3E_2E_1A = (E_3E_2E_1)A = I$ at the end, that means $(E_3E_2E_1) = A^{-1}$. That is why the right side of the augmented matrix becomes A^{-1} at the end of the process.

This is true in general, that is, multiplying a matrix A by an elementary matrix E from the left side of A has the same effect as performing the same row operation used to obtain E on A. Thus we can generalize the above example to any matrix A: if we can perform a sequence of elementary row operations on A to reduce it to I, and the same sequence of elementary row operations change the identity matrix to B, then it means that $(E_kE_{k-1}\cdots E_2E_1)A = I$, where E_1 is the elementary matrix corresponds to the first elementary row operation we used, E_2 is the elementary matrix corresponds to the second elementary row operation we used, and so on, E_k is the elementary matrix corresponds to the last elementary row operation we used. So $E_kE_{k-1}\cdots E_2E_1 = A^{-1}$ and $E_kE_{k-1}\cdots E_2E_1$ is precisely the matrix at the right side of the augmented matrix at the end of the process.

One more terminology: We say that two matrices A, B are row-equivalent if B is obtained from A by applying a sequence of row operations to A. That is, $B = E_k E_{k-1} \cdots E_2 E_1 A$ for some elementary matrices E_1, E_2, ..., E_k. For example, a matrix A and its echelon form are row equivalent, and A and its reduced row echelon form are also row equivalent. And, a square matrix A is invertible also means A is row equivalent to I.

An elementary matrix is always invertible: We saw that we can use one elementary row operation to change an elementary matrix to the identity. That is, if E is an elementary matrix, then there is another elementary matrix E' such that $E'E = I$. So E is invertible. In fact we know more.

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Quiz Question 3. Let E be the elementary matrix obtained by applying the row operation $3R_2 + R_1 \longrightarrow R_1$, then which of the following is E^{-1}?

A. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 2.6.3: Invertible Matrix Theorem.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Let A be an $n \times n$ square matrix. Then the following statements are equivalent.

1. A is invertible.
2. A is row equivalent to the $n \times n$ identity matrix I_n.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
5. An $n \times n$ matrix C exists such that $CA = I_n$.
6. An $n \times n$ matrix D exists such that $AD = I_n$.
7. The transpose matrix A^T is invertible.

This means that if A is invertible, then any of the statements 2–7 is also true. On the other hand, if any of the statements 2–7 is true, then A is invertible.

With what we have learned, we can in fact prove the theorem. For example, if A is invertible, then it is obvious that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (namely the zero solution) $\mathbf{x} = \mathbf{0}$, we can see this by multiplying A^{-1} on both side of the equation (from left). How do we argue the other way around, that is, if we know the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, how can prove that A must be invertible?

Quiz Question 4. If A is a 3×3 matrix and the following three row operations on A results in the identity matrix:

Step 1. $R_1 \longleftrightarrow R_3$; E_1

Step 2. $-2R_1 + R_2 \longrightarrow R_2$; E_2

Step 3. $R_2 + R_3 \longrightarrow R_3$. E_3

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_3 E_2 E_1 A$$

Which of the following is A^{-1}?

A. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$