Number Systems I

Number Representation

- Decimal Notation: 10 digits (0-9)
- Binary Notation: 2 digits (0,1)
- 10 is the base for Decimal, 2 is the base for binary notation

1 is at position 3

3 is at position 2

0 is at position 1

8 is at position 0

$$1308 = 1\times10^3 + 3\times10^2 + 0\times10^1 + 8\times10^0$$

10 is the base

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$$\mathbf{1010} = \mathbf{1} \times 2^{3} + \mathbf{0} \times 2^{2} + \mathbf{1} \times 2^{1} + \mathbf{0} \times 2^{0}$$

2 is the base

Number Representation

- Representation of n base b is called the base b expansion of n

For an integer b > 1. Every positive integer n can be expressed uniquely as:

$$n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots a_1 \cdot b^1 + a_0 \cdot b^0,$$

where k is a non-negative integer, each a_i is an integer in the range from 0 to b - 1, and $a_k \neq 0$.

$$\mathbf{1308} = \mathbf{1} \times 10^3 + \mathbf{3} \times 10^2 + \mathbf{0} \times 10^1 + \mathbf{8} \times 10^0$$

Converting from base b to decimal

For an integer b > 1. Every positive integer n can be expressed uniquely as:

$$n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots a_1 \cdot b^1 + a_0 \cdot b^0,$$

where k is a non-negative integer, each a_i is an integer in the range from 0 to b - 1, and $a_k \neq 0$.

$$(2012)_3 = 2 \times 3^3 + 0 \times 3^2 + 1 \times 3^1 + 2 \times 3^0$$

$$= 2 \times 27 + 0 \times 9 + 1 \times 3 + 2 \times 1$$

$$= 59$$

Hexadecimal Numbers

- 16 digits
- 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Hex	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Binary	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

- A 4-bit binary number can be represented with 1 Hexadecimal Digit
- 1 byte is equivalent to 8 bits
- So, 1 byte can be represented by 2 Hexadecimal digits

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Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Binary	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

$$(3B2)_{16} = 3 \times 16^2 + B \times 16^1 + 2 \times 16^0$$

$$= 3 \times 256 + 11 \times 16 + 2 \times 1$$

$$= 946$$

Converting from decimal to base b

For an integer b > 1. Every positive integer n can be expressed uniquely as:

$$n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots a_1 \cdot b^1 + a_0 \cdot b^0,$$

where k is a non-negative integer, each a_i is an integer in the range from 0 to b - 1, and $a_k \neq 0$.

[Base b expansion of (n div b)] [n mod b]

Converting from decimal to base b

Given n = 1161

$1161 = (3246)_7$

Find base 7 digits:

$3 \cdot 7^3 + 2 \cdot 7^2 + 4 \cdot 7 + 6 = 1161$

Digit in the range 0, 1, ..., 6

Multiple of 7

$3 \cdot 7^3 + 2 \cdot 7^2 + 4 \cdot 7 + 6 = 1161$

Rightmost digit is 1161 mod 7 = 6

For remaining digits use 1161 div 7 = 165

$3 \cdot 7^2 + 2 \cdot 7 + 4 \quad 6$

$3 \cdot 7 + 2 \quad 4$

Next digit is 165 mod 7=4

For remaining digits use 165 div 7= 23

$3 \quad 2$

Next digit is 23 mod 7=2

For remaining digits use 23 div 7=3

Since 3 is in the range 0, 1,...,6 → All Done!

1161 = $(3\,2\,4\,6\,)_7$

Example

- Convert the decimal number 542 to a base 5 number.

Number of digits to represent a number

- Given an integer n, the number of digits required to represent n in base b is:

$$\lceil \log_b(n+1) \rceil$$

- Example: how many digits are required to express the base 2 expansion of 13?

$$\lceil \log_2(13+1) \rceil = \lceil \log_2(14) \rceil = \lceil 3.8 \rceil = 4$$